MATHEMATICS

magazine

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THE EDITOR'S PAGE

PUBLISH OR PERISH

Recently a college professor in one of our prominent institutions of higher learning resigned from his position, stating that promotions were based upon evaluation of published works rather than upon demonstrated teaching ability. The local press immediately placed the issue on the front page and criticised what it called the policy of "Publish or Perish" in faculty promotion. We doubt that many first rate universities have such a narrow promotion policy and certainly feel that none should have such a policy.

If we were called upon to coin a name for a more suitable promotion policy for college faculties, it would be "Communicate or Quit." It is certainly the responsibility of the effective professor to communicate his knowledge to others, either through scholarly publications, personal contact with colleagues and graduate students, or in lectures to classes of undergraduates. A better criterion by which to measure the professor is how well he performs one or all of these types of communication. Even skillful research is of little value until it is communicated to others.

MOVING?

Have you moved recently? It is surprising how many mathematicians are on the move. The MATHEMATICS MAGAZINE is mailed by second class postage and will not be forwarded by the Post Office to your new address. If you wish to receive your magazine without delay, please inform us of your correct new address. By the way, did you note that the MATHEMATICS MAGAZINE also has a new address? Please use it, and we can serve you better.

R.E.H.

SOME REFLECTIVE GEOMETRY OF THE TRIANGLE

D. Moody Bailey

INTRODUCTION

If BC be the perpendicular bisector of XX', we shall say that $X'(X)$ is the reflection of $X(X')$ with respect to BC. Similarly, when CA and AB are the perpendicular bisectors of YY' and ZZ' respectively, we say that $Y'(Y)$ is the reflection of $Y(Y')$ with respect to CA and $Z'(Z)$ is the reflection of $Z(Z')$ with respect to AB. Suppose ABC and XYZ be two triangles in the plane. Reflect X with respect to BC, Y with respect to CA, and Z with respect to AB thereby obtaining triangle $X'Y'Z'$. Under what conditions will triangles XYZ and $X'Y'Z'$ be similar or as we shall say "reflective" with respect to triangle ABC? Will the similarity be direct or inverse? If triangle XYZ is "reflective" with respect to triangle ABC, is triangle ABC "reflective" with respect to triangle XYZ?

These and other questions are answered in the discussion to follow. "Reflectivity" is shown to be reciprocal and furthermore the "reflective" pairs of triangles are shown to occur in groups of fours. Many important facts of "reflectivity" are omitted in the present article with the hope that they may be considered at a future time.

Three circles in the plane have point P in common and their other intersections form triangle ABC (Fig. 1). A fourth circle through P meets
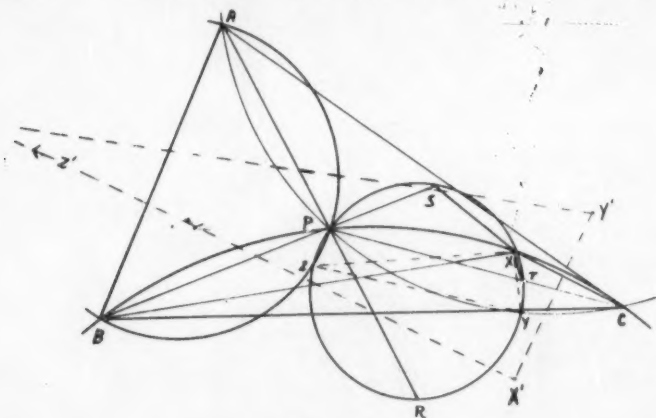
Figure 1.

the three original circles BPC, CPA, and APB in points X, Y, and Z respectively. Extend BP and CP to meet the fourth circle in points S and T

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and determine the value of BX/XC. Considering triangles BSX and CTX, we have $\angle XBS = \angle XCT$, since both angles are inscribed in circle BPC and each is measured by one-half arc PX. Also, $\angle BSX$ or $\angle PSX$ is supplementary to $\angle PTX$ since the points P, S, X, and T lie on the fourth circle and the angles at S and T are opposite angles in the cyclic quadrilateral $PSXT$. $\angle CTX$ is also supplementary to $\angle PTX$ and we have $\angle BSX = \angle CTX$. Two angles of triangle BSX are then equal to two angles of triangle CTX. As a result the third angles of the two triangles must be equal and triangles BSX and CTX are similar giving $(BX/XC) = (BS/CT)$. If we now extend AP to meet the fourth circle at R, we shall be able to show in the same manner as above that $(CY/YA) = (CT/AR)$ and that $(AZ/ZB) = (AR/BS)$. We then have $(BX/XC) \cdot (CY/YA) \cdot (AZ/ZB) = (BS/CT) \cdot (CT/AR) \cdot (AR/BS) = 1$. Accordingly, the following fundamental result:

Theorem 1. Three circles have point P in common and their other intersections form triangle ABC. A fourth circle through P meets circles BPC, CPA, and APB in points X, Y, and Z respectively. Then $(BX/XC) \cdot (CY/YA) \cdot (AZ/ZB) = 1$.

This seems to be a rather important result and we immediately wonder whether the converse may be true. That is, if we select points X, Y, and Z on circles BPC, CPA, and APB so that $(BX/XC) \cdot (CY/YA) \cdot (AZ/ZB) = 1$, have we thus determined a circle XYZ that always passes through point P? Unfortunately, or so it now seems, such is not the case. For example, let us take points X, Y, and Z lying without triangle ABC and on circles BPC, CPA, and APB in a manner such that $BX = XC$, $CY = YA$, and $AZ = ZB$ (Fig. 2). Obviously $(BX/XC) \cdot (CY/YA) \cdot (AZ/ZB) = 1$ and equally obviously circle XYZ does not pass through point P. On the other hand it seems that the diametric opposites of X, Y, and Z on circles BPC, CPA, and APB might determine a circle through this point.
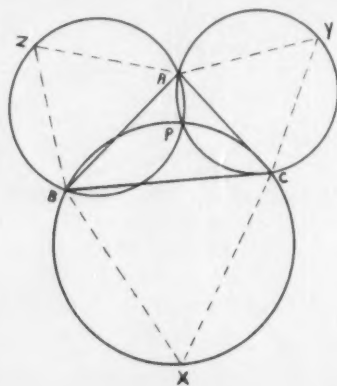
Figure 2

To shed further light on the matter and to aid in establishing conditions necessary for the statement of its converse, let us prove theorem 1

by another method. Suppose that a straight line meets the sides of triangle $A''B''C''$ in points X'', Y'', and Z'' (Fig. 3). Using the theorem of Menelaus, without assigning signed values to the segments involved, we obtain
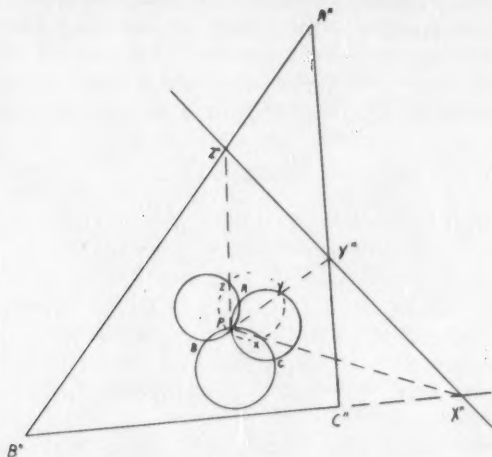
Figure 3.

$(B''X''/X''C'') \cdot (C''Y''/Y''A'') \cdot (A''Z''/Z''B'') = 1$. Now choose P, any point in the plane of triangle $A''B''C''$, and invert the sides of the triangle and line $X''Y''Z''$ with respect to point P. The triangle sides $B''C''$, $C''A''$, and $A''B''$ invert into three circles through P that have for their other common points A, B, and C. This is true because the inverse of a straight line is a circle through the center of inversion. Let us next invert line $X''Y''Z''$ with respect to point P and we get a fourth circle through P that meets the other three circles at points X, Y, and Z. Now the length of the inverse segment BX, where k is the constant of inversion, is given by the expression $BX = (k \cdot B''X'')/(PB'' \cdot PX'')$. This is a well known theorem of modern geometry and is not difficult to establish. Likewise, $XC = (k \cdot X''C'')/(PX'' \cdot PC'')$ from which $(BX/XC) = (B''X''/X''C'') \cdot (PC''/PB'')$. In a similar manner $(CY/YA) = (C''Y''/Y''A'') \cdot (PA''/PC'')$ and $(AZ/ZB) = (A''Z''/Z''B'') \cdot (PB''/PA'')$. This gives

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{B''X''}{X''C''} \cdot \frac{C''Y''}{Y''A''} \cdot \frac{A''Z''}{Z''B''} \cdot \frac{PC''}{PB''} \cdot \frac{PA''}{PC''} \cdot \frac{PB''}{PA''}$$

$$= \frac{B''X''}{X''C''} \cdot \frac{C''Y''}{Y''A''} \cdot \frac{A''Z''}{Z''B''}$$

which equals unity, as we have seen. Therefore points X, Y, and Z again lie on a fourth circle through P and $(BX/XC) \cdot (CY/YA) \cdot (AZ/ZB) = 1$.

Let us now notice that the transversal $X''Y''Z''$ may cut the sides of triangle $A''B''C''$ in a couple of ways. All three sides of triangle $A''B''C''$

may be cut externally or one side may be cut externally and the other two internally. What significance does this have in our inverted figure?

Consider any line L and P a point in the plane with respect to which L is to be inverted. Suppose A'' and B'' be two fixed points on line L and X'' a variable point thereon. Invert L with respect to P obtaining a circle through P whose center lies on the perpendicular from P to line L. Then A lies on PA'' and B lies on PB''. Suppose point X'' has a position on L external to the segment $A''B''$ (Fig. 4A). It is then evident that X and P will
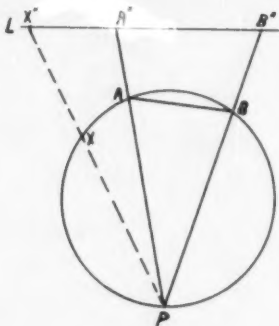
Figure 4A.

both lie on the same side of chord AB. If X'' lies between A'' and B'', it is further evident that X and P lie on opposite sides of chord AB (Fig. 4B).
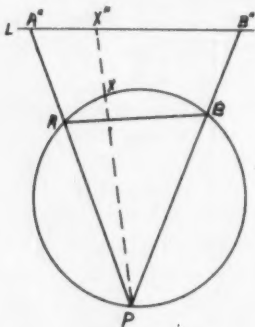
Figure 4B.

These observations are of great importance since they shall shortly allow us to express the converse of theorem 1.

We return to Fig. 3 and apply these results. If here, line $X''Y''Z''$ cuts the three sides of triangle $A''B''C''$ externally, we shall know that P and X are on the same side of BC. Likewise, P and Y will be on the same side of CA, and P and Z will be on the same side of AB. If the other

possibility exists; namely, that line $X''Y''Z''$ cuts one side of triangle $A''B''C''$ externally and the other two sides internally, then one of the points X, Y, or Z will lie with P on the same side of the corresponding side of triangle ABC while the other two points will lie opposite to P with respect to their sides of triangle ABC. We may now express the converse of theorem 1.

Theorem 2A. Three points X, Y, and Z are chosen on circles BPC, CPA, and APB respectively such that P and X lie on the same side of BC, P and Y lie on the same side of CA, and P and Z lie on the same side of AB. If $(BX/XC) \cdot (CY/YA) \cdot (AZ/ZB) = 1$, then will the four points X, Y, Z, and P be cyclic. Again, points X, Y, and Z are chosen on circles BPC, CPA, and APB so that P and (X, Y, Z) lie on the same side of $(BC, CA,$ and $AB)$, P and (Y, Z, X) lie on opposite sides of (CA, AB, BC), and P and (Z, X, Y) lie on opposite sides of (AB, BC, CA). If $(BX/XC) \cdot (CY/YA) \cdot (AZ/ZB) = 1$, the four points X, Y, Z, and P will be cyclic.

It is therefore necessary, when dealing with the converse theorem, to know the position of points X, Y, Z, and P with respect to the sides of triangle ABC before we can assert for certainty that the four points are cyclic. Returning to Fig. 2, we now understand why points X, Y, and Z do not determine a circle through P. P and X are on opposite sides of BC, P and Y are on opposite sides of CA, and P and Z are on opposite sides of AB. It is now evident that the diametric opposites of X, Y, and Z on circles BPC, CPA, and APB will determine a circle through P. Also the diametric opposite of X on circle BPC will lie on circle PYZ, the diametric opposite of Y on circle CPA will lie on circle PZX, and the diametric opposite of Z on circle APB will lie on circle PXY.

Theorem 2A is of undue length and we search for a way to shorten it. Let us give signs to the ratios (BX/XC), (CY/YA), and (AZ/ZB). We agree to call the ratio (BX/XC) positive if P and X lie on the same side of BC, negative if P and X lie on opposite sides of BC. Similarly for the ratios (CY/YA) and (AZ/ZB). When this is done we can assert for certainty that the selection of points X, Y, and Z on circles BPC, CPA, and APB so that $(BX/XC) \cdot (CY/YA) \cdot (AZ/ZB) = 1$ means always that points X, Y, Z, and P are cyclic. This follows from the fact that under the conditions of theorem 2A all of the three ratios will be positive or one of them will be positive and the other two will be negative thus guaranteeing that the product of the three ratios will be equal to positive unity. We then give signed values to the ratios (BX/XC), (CY/YA), and (AZ/ZB) and express theorem 2A in a more compact form.

Theorem 2B. Three points X, Y, and Z are chosen on circles BPC, CPA, and APB respectively so that $(BX/XC) \cdot (CY/YA) \cdot (AZ/ZB) = 1$. Then will the four points X, Y, Z, and P be cyclic.

Theorems 1 and 2 are of great importance in the study of the geometry of the triangle. The reader has probably already observed that they may both be obtained by inverting the theorem of Menelaus and its converse. The author would consider theorems 1 and 2 fully as important as the two

well known theorems from which they are obtained through the process of inversion. We hope to justify this statement as our study progresses.

We next endeavor to determine the value of the angles of triangle XYZ. Considering Fig. 1, we have $\sphericalangle ZXY = \sphericalangle ZPY$ since the points Z, P, X, and Y are on the fourth circle through P. Also $\sphericalangle ZPY = \sphericalangle ZPR + \sphericalangle RPY$. Now $\sphericalangle ZPR = \sphericalangle ABZ$ since the points A, P, Z, and B lie on circle APB. In the same manner $\sphericalangle RPY = \sphericalangle ACY$ since points A, P, Y, and C lie on circle CPA. Returning to $\sphericalangle ZPY$, we thus have $\sphericalangle ZPY = \sphericalangle ZPR + \sphericalangle RPY = \sphericalangle ABZ + \sphericalangle ACY$ or $\sphericalangle ZXY = \sphericalangle ABZ + \sphericalangle ACY$. In a similar fashion we show that $\sphericalangle XYZ = \sphericalangle BCX + \sphericalangle BAZ$ and $\sphericalangle YZX = \sphericalangle CAY + \sphericalangle CBX$.

Before expressing our fundamental angle theorem we must note that for some cases of Fig. 1, $\sphericalangle ZXY$ may be equivalent to the difference of $\sphericalangle$'s ABZ and ACY rather than to their sum. Similarly for $\sphericalangle$'s XYZ and YZX. Since a full consideration of the various cases involving the sum or difference of $\sphericalangle$'s ABZ and ACY, etc., requires quite a little time and space we shall omit same at this point. However, we should state that either all the angles of triangle XYZ will be expressed as the sum of two angles or one of them will be so expressed while the other two will be equivalent to the difference of two angles. Should the value of an angle of triangle XYZ involve a difference, we should be careful to express this difference in the correct form. That is, if $\sphericalangle ZXY$ should involve the difference of $\sphericalangle$'s ABZ and ACY, and if $\sphericalangle ACY$ should be larger than $\sphericalangle ABZ$, we would then have to change the value of $\sphericalangle ZXY$ given above to the value $\sphericalangle ZXY = \sphericalangle ACY - \sphericalangle ABZ$. Similarly for the other angle of triangle XYZ that involves a difference. With this in mind, and denoting the angles of triangle XYZ with the letters X, Y, and Z, we have

Theorem 3. Circles BPC, CPA, and APB are met by a fourth circle through P in points X, Y, and Z respectively. Then $\sphericalangle X = \sphericalangle ABZ$ plus or minus $\sphericalangle ACY$, $\sphericalangle Y = \sphericalangle BCX$ plus or minus $\sphericalangle BAZ$, and $\sphericalangle Z = \sphericalangle CAY$ plus or minus $\sphericalangle CBX$.

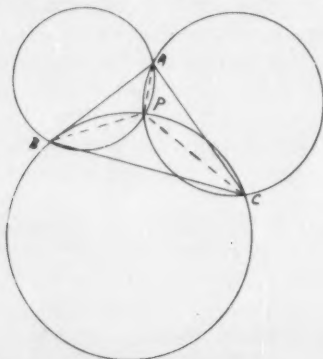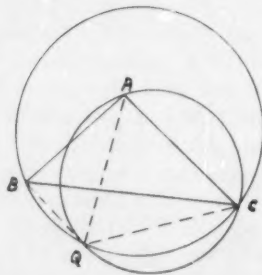We proceed to a consideration of Fig's. 5A and 5B. Here three circles

Figure 5A.   Figure 5B.

in the plane have point P in common and their other intersections form triangle ABC (Fig. 5A). In this instance let P be a point lying within triangle ABC so that $\angle BPC + \angle CPA + \angle APB = 360°$. Imagine that circle BPC is rotated about side BC through an angle of $180°$ until arc BPC falls on the opposite side of BC. The new position of circle BPC will be shown in Fig. 5B. When this operation is completed we shall say that circle BPC has been "reflected" with respect to BC. In a similar manner let circle CPA be reflected about CA, and let Q be the point of intersection (other than C) of these two reflected circles (Fig. 5B). Considering Fig's. 5A and 5B, we see that $\angle BQC = \angle BPC$ and $\angle CQA = 180° - \angle CPA$. Now $\angle AQB = \angle BQC - \angle CQA = \angle BPC - (180° - \angle CPA) = (\angle BPC + \angle CPA) - 180° = (360° - \angle APB) - 180° = 180° - \angle APB$. This value for $\angle AQB$ shows that when circle APB is reflected about AB, the reflected circle must pass through point Q. A full consideration of all cases involved leads to

Theorem 4. Three circles have point P in common and their other points of intersection form triangle ABC. Circles BPC, CPA, and APB are reflected about sides BC, CA, and AB respectively. The reflected circles thus obtained have a point Q in common.

We shall speak of points P and Q as "reflective points" with respect to triangle ABC since when one is given the other may be determined through the reflection of a triad of circles about the sides of triangle ABC. Points P and Q are of equal significance. That is, if circles BQC, CQA, and AQB are the original circles having Q as their common point, they reflect into circles BPC, CPA, and APB having P as their common point. Points P and Q are important points with respect to triangle ABC and we shall list some of their properties as we proceed. We should further observe that the angles formed by rays from P to the vertices of triangle ABC are either equal or supplementary to the angles formed by corresponding rays from Q to the vertices of triangle ABC. For example, in Fig's. 5A and 5B, $\angle BPC = \angle BQC$, $\angle CPA$ is supplementary to $\angle CQA$, and $\angle APB$ is supplementary to $\angle AQB$.

To shed further light on reflective points P and Q, let us consider Fig. 6A. Here R and S are a pair of inverse points with respect to circle ABC. If perpendiculars be dropped from R to the sides of triangle ABC we determine triangle MNO which is usually called the pedal triangle of point R with respect to triangle ABC. Quadrilateral $ROBM$ is cyclic whence $\angle OMR = \angle 1$ and quadrilateral $RMCN$ is cyclic giving $\angle RMN = \angle 2$. Thus $\angle OMN = \angle 1 + \angle 2$ and similar expressions may be obtained for the other two angles of triangle MNO. Now, if P be the isogonal conjugate of R in triangle ABC (Fig. 6B), we see that $\angle BPC = 180° - (\angle 1 + \angle 2) = 180° - \angle OMN$. We may then say that the angles of the pedal triangle of point R are either equal or supplementary to the angles determined by rays from P, the isogonal conjugate of R, to the vertices of triangle ABC.

Now, if R and S are inverse points with respect to triangle ABC, then the pedal triangles of R and S with respect to triangle ABC will be similar. This fact is proved in any standard text on modern geometry and will

Figure 6A

Figure 6B

not be reproduced here. From the above discussion, the isogonal conju-
gates P and Q, of the inverse points R and S, are such that rays from P
and Q to the vertices of triangle ABC determine angles that are either
equal or supplementary. Hence P and Q must be a pair of reflective points.
By reversing the reasoning process we arrive at this important result.

Theorem 5. The isogonal conjugates of a pair of reflective points P
and Q with respect to triangle ABC is a pair of inverse points R and S
with respect to triangle ABC. Conversely, the isogonal conjugates of a
pair of inverse points R and S with respect to triangle ABC is a pair of
reflective points P and Q with respect to triangle ABC.

Some additional properties of reflective points P and Q must now be
discussed. Let us divide the plane in which triangle ABC lies into seven
regions as indicated in Fig. 7. Region I is bounded by extended segments
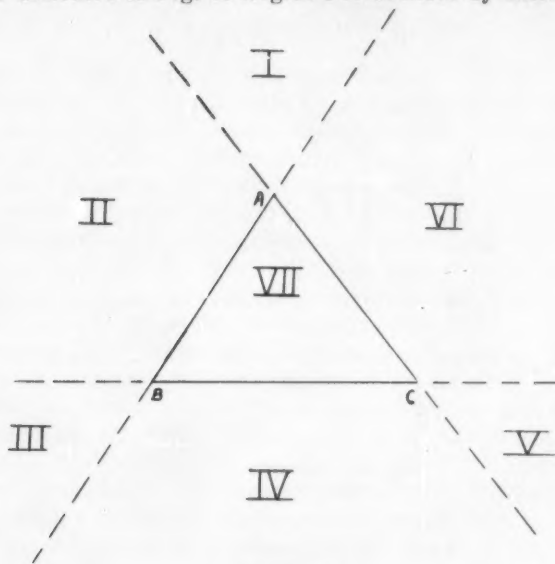
Figure 7

AB and CA, region II by AB and extended segments CA and BC, region III
by extended segments BC and AB, etc. Region VII lies within triangle
ABC and we make the following observations:

(a) For any point P within triangle ABC (region VII) $\angle BPC + \angle CPA + \angle APB = 360°$.

(b) For any point P without triangle ABC (regions I, II, III, IV, V, and
VI) one of the angles from P to two vertices of triangle ABC equals the
sum of the other two angles thus drawn. Thus, if P lies in regions I or IV,
$\angle BPC = \angle CPA + \angle APB$. If P lies in regions III or VI, $\angle CPA = \angle APB + \angle BPC$.
Finally, if P lies in regions II or V, $\angle APB = \angle BPC + \angle CPA$.

We shall now allow P to occupy positions in the various regions and
attempt to determine the position of its reflective partner Q. Let us first

allow P to be a point within triangle ABC (region VII) and study the various possible locations of point Q. As we do this let us turn our attention to Fig. 2 at the same time visualizing the seven regions of Fig. 7. We use Fig. 2 because there circle BPC is seen to consist of two arcs; namely, arc BPC within triangle ABC and arc BXC without triangle ABC. Likewise, circle CPA is seen to consist of arcs CPA and CYA, and circle APB consists of arcs APB and AZB. Now if P lies within triangle ABC (region VII) we have

(1) $$\sphericalangle BPC + \sphericalangle CPA + \sphericalangle APB = 360°.$$

Let us suppose that Q, the reflective partner of P with respect to triangle ABC, lies in region IV. For any point Q in region IV, whether it be the reflective partner of P or not, the equation

(2) $$\sphericalangle BQC = \sphericalangle CQA + \sphericalangle AQB$$

must hold. If Q is the reflective partner of P, then Q must lie on arc BPC after it is reflected about BC (Fig. 2) since arc BXC will lie wholly "above" side BC after circle BPC is reflected about BC. For this reason arc BXC cannot enter region IV. This then makes $\sphericalangle BQC = \sphericalangle BPC$. Q must lie on arc CYA of circle CPA since after the reflection of circle CPA about CA, arc CPA lies wholly to the "right" of side CA and no part of arc CPA can then lie in region IV. This being true gives $\sphericalangle CQA = 180° - \sphericalangle CPA$. Q must also lie on arc AZB of circle APB since after the reflection of circle APB about AB, arc APB lies wholly to the "left" of side AB and therefore no part of arc APB, after rotation, can lie in region IV. This means that $\sphericalangle AQB = 180° - \sphericalangle APB$. Returning to (2) and substituting values obtained we find that, if Q be the reflective partner of P and lie in region IV, then (2) $\sphericalangle BQC = \sphericalangle CQA + \sphericalangle AQB$ becomes $\sphericalangle BPC = (180° - \sphericalangle CPA) + (180° - \sphericalangle APB)$ which reduces to (1) $\sphericalangle BPC + \sphericalangle CPA + \sphericalangle APB = 360°$. The assumption that Q may lie in region IV when P is in region VII is therefore consistent with equations (1) and (2). Should this occur the important thing that we wish to emphasize is this: If points P and Q lie in regions VII and IV respectively then P and Q lie on the same side of segments CA and AB but on opposite sides of segment BC.

Proceeding as above we may further show that when P lies in region VII it is possible for Q to lie in region VI and if this occurs P and Q are on the same side of segments AB and BC but on opposite sides of segment CA. Finally, we may show that when P lies within triangle ABC (region VII) it is possible for Q to lie in region II. If this occurs P and Q will lie on the same side of segments BC and CA but on opposite sides of the segment AB.

If we take P in triangle ABC (region VII) and assume that Q lies in region I, we start with the fundamental equations (1) $\sphericalangle BPC + \sphericalangle CPA + \sphericalangle APB = 360°$ and (2) $\sphericalangle BQC = \sphericalangle CQA + \sphericalangle AQB$. By studying the reflected arcs we find that the angles to be made at Q are not consistent with equations (1) and (2) which leads to the conclusion that Q cannot lie in region I when

P lies in region VII. In the same way we show that Q cannot lie in regions III or V when P lies in region VII.

To sum it all up we say that when P lies in region VII its reflective partner Q must lie in either regions II, IV, or VI and as we have seen this means that P and Q will lie on the same side of two of the segments or sides of triangle ABC but will lie on opposite sides of one of the three segments of triangle ABC.

If we take P in region I and study angle values as above we find that Q may lie in regions II, IV, or VI. If P lies in region I and Q lies in regions II or VI, we again have P and Q lying on the same side of two of the segments of the triangle but on opposite sides of the third segment. However, if P lies in region I and Q lies in region IV, a change occurs and we find that P and Q lie on opposite sides of each of the three segments of triangle ABC. It is possible to demonstrate that when P lies in region I, Q cannot lie in regions III, V, or VII. Similar results may be obtained for regions III and V and we are led to the conclusion that if P lies in regions I, III, or V then Q must also lie without triangle ABC and we accordingly have both points P and Q without triangle ABC. This conclusion may also be reached through an analysis of theorem 5.

If we take point P in region IV, we may show that it is possible for Q to lie in regions I, III, V, or VII. Without discussing the other regions in which P may be chosen and the regions in which its partner Q may lie, we shall simply state that when they are all duly considered a remarkable fact emerges which is of extreme importance in building a secure foundation for the work to follow. Speaking of the sides BC, CA, and AB of triangle ABC as segments we state the conclusion in this manner:

Theorem 6. If P and Q are a pair of reflective points with respect to triangle ABC, then (1) P and Q lie on the same side of two segments of triangle ABC while they lie on opposite sides of the third segment, or (2) P and Q lie on opposite sides of all three segments of triangle ABC.

Let us now reflect point X with respect to side BC obtaining point X', point Y with respect to CA obtaining Y', and point Z with respect to AB obtaining Z'. It is to be noted that points X and X' are not reflective points as were points P and Q. Rather BC is the perpendicular bisector of XX' (Fig. 1). Also CA is the perpendicular bisector of YY' and AB is the perpendicular bisector of ZZ'. Point X' lies on circle BPC reflected with respect to BC, point Y' lies on circle CPA reflected with respect to CA, and point Z' lies on circle APB reflected with respect to AB. Therefore points X', Y', and Z' lie on the circles passing through point Q (theorem 4). Obviously $BX = BX'$, $XC = X'C$, $CY = CY'$, $YA = Y'A$, $AZ = AZ'$, and $ZB = Z'B$. Thus $(BX'/X'C) = (BX/XC)$, $(CY'/Y'A) = (CY/YA)$, and $(AZ'/Z'B) = (AZ/ZB)$. Now $(BX/XC) \cdot (CY/YA) \cdot (AZ/ZB) = 1$ by theorem 1 and consequently $(BX'/X'C) \cdot (CY'/Y'A) \cdot (AZ'/Z'B) = 1$. But, is it possible to show that the conditions of theorem 2 are satisfied for the points X', Y', and Z' or may points X', Y', and Z' play the same role as did points X, Y, and Z in Fig. 2? If so, circle $X'Y'Z'$ will not pass through

point Q. Fortunately, at this point, we are able to enlist the aid of theorem 6.

We know that $(BX/XC) \cdot (CY/YA) \cdot (AZ/ZB) = 1$ when signed values are given to the ratios (BX/XC), (CY/YA), and (AZ/ZB) because circle XYZ passes through point P. Can we determine the signs of ratios $(BX'/X'C)$, $(CY'/Y'A)$, and $(AZ'/Z'B)$ by comparing them with ratios (BX/XC), and (AZ/ZB)? In the first place we know that X', Y', and Z' lie opposite to X, Y, and Z with respect to BC, CA, and AB since they are obtained by reflecting X, Y, and Z with respect to these sides. By theorem 6 part (1) Q will be opposite to P with respect to one segment or side of triangle ABC. Should (1) then occur we see that the signs of two of the ratios $(BX'/X'C)$, $(CY'/Y'A)$, and $(AZ'/Z'B)$ will be different from the signs of the corresponding ratios (BX/XC), (CY/YA), and (AZ/ZB). Since the signed ratios (BX/XC), (CY/YA), and (AZ/ZB) fulfill the conditions of theorem 2B, it is clear that a change in the signs of two of them will still permit the product of the three to equal positive unity. Accordingly, when P and Q satisfy the conditions of part (1) of theorem 6, we have $(BX'/X'C) \cdot (CY'/Y'A) \cdot (AZ'/Z'B) = 1$ when signed values are given to the ratios. This means that circle $X'Y'Z'$ must pass through point Q.

Suppose P and Q satisfy the conditions of theorem 6 part (2). Then Q is opposite to P with respect to all sides or segments of triangle ABC. Accordingly, if P and X have a certain relationship to side BC (say both on the same side of BC), then Q and X' will have the same relationship to side BC (both on the same side of BC — even though it be the other side of BC as compared to P and X). Thus the sign of $(BX'/X'C)$ will be the same as the sign of (BX/XC). Likewise $(CY'/Y'A)$ and $(AZ'/Z'B)$ will have the same signs as do (CY/YA) and (AZ/ZB). As no change in signs occur, we now know that if $(BX/XC) \cdot (CY/YA) \cdot (AZ/ZB) = 1$, then $(BX'/X'C) \cdot (CY'/Y'C) \cdot (AZ'/Z'B) = 1$ when signed values are given to all ratios. Accordingly, when P and Q satisfy the conditions of part (2) theorem 6, circle $X'Y'Z'$ must still pass through point Q. Thus, for all possibilities, we have:

Theorem 7A. Three circles have point P in common and their other intersections form triangle ABC. A fourth circle through P meets circle BPC, CPA, and APB in points X, Y, and Z respectively. Points X, Y, and Z are reflected with respect to sides BC, CA, and AB obtaining points X', Y', and Z' respectively. Circle $X'Y'Z'$ passes through point Q, the reflective partner of point P.

The results of the above discussion may be presented in more spectacular form as follows:

Theorem 7B. A variable circle through P meets fixed circles BPC, CPA, and APB in points X, Y, and Z respectively. X, Y, and Z are reflected with respect to BC, CA, and AB determining points X', Y', and Z'. Variable circle $X'Y'Z'$ always passes through fixed point Q, the reflective partner of point P with respect to triangle ABC.

If P and R be a pair of isogonal conjugates in triangle ABC, then

their pedal triangles DEF and MNO have a common circumcircle whose center lies at L, the midpoint of PR (Fig. 8). This is another well known theorem found in texts on modern geometry. With P as center let us multiply D, E, and F by 2 obtaining points P_1', P_2', and P_3' which are seen to be the reflections of P with respect to sides BC, CA, and AB of triangle ABC. Also the center of the circumcircle of triangle $P_1'P_2'P_3'$ is seen to be R since the center of circle DEF was at L, the midpoint of PR. Now point
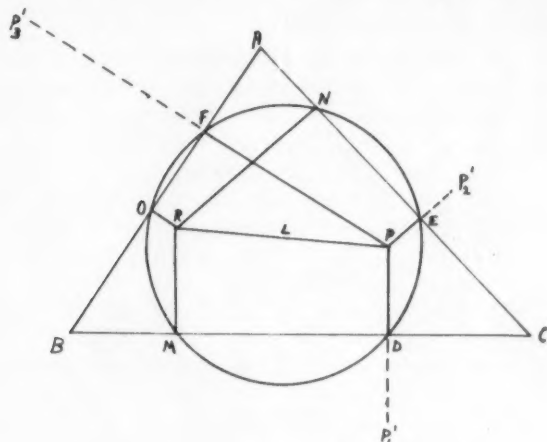
Figure 8

P may be regarded as the null circle XYZ passing through point P and its reflections about sides BC, CA, and AB give points P_1', P_2', and P_3' respectively. Circle $P_1'P_2'P_3'$ must therefore pass through point Q (theorem 7A). This yields

Theorem 8. Any point P is reflected about sides BC, CA, and AB of triangle ABC determining points P_1', P_2', and P_3' respectively. The center of circle $P_1'P_2'P_3'$ is at R, the isogonal conjugate of P in triangle ABC, and circle $P_1'P_2'P_3'$ passes through Q, the reflective partner of P in triangle ABC.

Obviously the same result holds for point Q and we may combine the two results, together with some help from theorem 5, and make this statement:

Theorem 9. P and Q, a pair of reflective points in triangle ABC, are reflected with respect to sides BC, CA, and AB of triangle ABC obtaining triangles $P_1'P_2'P_3'$ and $Q_1'Q_2'Q_3'$ respectively. The centers of circles $P_1'P_2'P_3'$ and $Q_1'Q_2'Q_3'$ are points R and S, the isogonal conjugates of P and Q respectively in triangle ABC. R and S are also a pair of inverse points with respect to circle ABC.

P is the homothetic center of circles DEF and $P_1' P_2' P_3'$ (Fig. 8) and the homothetic ratio is 2. We have seen that circle $P_1' P_2' P_3'$ passes through Q (theorem 8). Homology shows that circle DEF must pass through the midpoint of PQ and we have

Theorem 10. DEF is the pedal triangle of any point P with respect to triangle ABC. Circle DEF passes through the midpoint of PQ, where P and Q are a pair of reflective points with respect to triangle ABC.

Suppose that P is any point in the plane of triangle ABC and that A_1, B_1, and C_1 are the midpoints of sides BC, CA, and AB of triangle ABC (Fig. 9). With P as center of similitude multiply triangle $A_1 B_1 C_1$ by 2 obtaining points H, I, and J respectively. Now $BHCP$ is seen to be a parallelogram whose diagonals are bisected at A_1. Therefore $(BH/HC) = (CP/PB)$
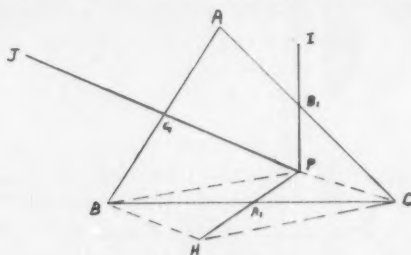
Figure 9

and in similar fashion $(CI/IA) = (AP/PC)$ and $(AJ/JB) = (BP/PA)$. So $(BH/HC) \cdot (CI/IA) \cdot (AJ/JB) = (CP/PB) \cdot (AP/PC) \cdot (BP/PA) = 1$. Since $BHCP$ is a parallelogram $\sphericalangle BHC = \sphericalangle BPC$. This means that H lies on circle BPC after it is reflected about side BC. In other words H lies on circle BQC and in similar manner I and J lie on circles CQA and AQB respectively. Suppose we were to now construct a circle through P whose center lies at the circumcenter of triangle ABC. If we then reflect the points in which this circle meets circles BPC, CPA, and APB about sides BC, CA, and AB it is evident that we obtain points H, I, and J. This allows us to now assert that circle HIJ passes through Q (theorem 7A). We further note that triangle HIJ is similar to triangle $A_1 B_1 C_1$ and therefore similar to triangle ABC. In fact triangle HIJ is congruent to triangle ABC since the sides of each are double those of triangle $A_1 B_1 C_1$. Since circle HIJ passes through point Q, we see that circle $A_1 B_1 C_1$ must pass through the midpoint of PQ. This is true because P is the homothetic center of triangles HIJ and $A_1 B_1 C_1$ with the homothetic ratio being 1/2. However, circle $A_1 B_1 C_1$ is the nine-point circle of triangle ABC and we have just arrived at one of the most remarkable theorems of the lot.

Theorem 11. P and Q are a pair of reflective points with respect to triangle ABC. The midpoint of PQ lies on the nine-point circle of triangle

ABC.

Should point P of theorem 11 be the orthocenter of triangle ABC, circles BPC, CPA, and APB are then equal. This is another well known theorem of modern geometry. If we then reflect circles BPC, CPA, and APB about sides BC, CA, and AB of triangle ABC each reflected circle becomes the circumcircle of triangle ABC. This means that point Q may be any point on the circumcircle of triangle ABC. Thus, when P is the orthocenter of triangle ABC, it has an infinity of reflective partners. In this particular instance the truth of theorem 11 is apparent because another theorem found in geometrical texts states that a line joining the orthocenter of a triangle to any point on the circumcircle is bisected by the nine-point circle of the triangle.

We have shown that the circumcircle of triangle DEF, the pedal triangle of point P with respect to triangle ABC, passes through the midpoint of PQ (theorem 10). Since the nine-point circle of triangle ABC also goes through this point (theorem 11), we have this point common to the nine-point circle and to the circumcircle of triangle DEF. As we have seen (Fig. 8) the pedal circle of point P is identical with the pedal circle of point R, the isogonal conjugate of point P. If we should locate the reflective partner of point R, say T, then circle MNO (DEF) would pass through the midpoint of RT (theorem 10) and the nine-point circle of triangle ABC would also go through this point (theorem 11). Therefore, the one pedal circle DEF (MNO) would meet the nine-point circle of triangle ABC in two points; namely, the midpoints of PQ and RT. Suppose we were to let point P become the incenter of triangle ABC. Then points P and R would become identical and the lines PQ and RT would coincide. This means that the incircle and nine-point circle of triangle ABC would have only one point in common; namely, the midpoint of the line connecting the incenter with its reflective partner. The same reasoning applies to the three excenters of triangle ABC and we rather easily arrive at the famous theorem of Feuerbach; that is, the nine-point circle of a triangle is tangent to the inscribed and to each of the escribed circles of the triangle.

We have reflected X with respect to BC, Y with respect to CA, and Z with respect to AB obtaining triangle $X'Y'Z'$ (Fig. 1). Furthermore we have shown that circle $X'Y'Z'$ passes through point Q, the point common to reflected circles BPC, CPA, and APB (theorem 7). Accordingly, we may apply the results of theorem 3 to the angles of triangle $X'Y'Z'$ and have $\angle X' = \angle ABZ'$ plus or minus $\angle ACY'$, $\angle Y' = \angle BCX'$ plus or minus $\angle BAZ'$, and $\angle Z' = \angle CAY'$ plus or minus $\angle CBX'$. As X' is obtained by reflecting X with respect to BC, it is easily seen that $\angle BCX' = \angle BCX$ and that $\angle CBX' = \angle CBX$. Likewise, $\angle CAY' = \angle CAY$, $\angle ACY' = \angle ACY$, $\angle ABZ' = \angle ABZ$, and $\angle BAZ' = \angle BAZ$. From these equalities we see that the angles of triangle XYZ (theorem 3) are equal to the angles of triangle $X'Y'Z'$. The similarity of triangles XYZ and $X'Y'Z'$ turns out to be inverse rather than direct. We shall not record the proof of inverse similarity as this article has already become rather extended. We should say that being inversely

similar means that triangles XYZ and $X'Y'Z'$ cannot be made to coincide through an homology until one of them is reflected about a line in their plane through an angle of 180°. We now record our most important result:

Theorem 12. Circles BPC, CPA, and APB are met by a fourth circle through P in points X, Y, and Z respectively. Points X, Y, and Z are then reflected with respect to BC, CA, and AB obtaining points X', Y', and Z' respectively. Triangles XYZ and $X'Y'Z'$ are inversely similar.

At this point the question may arise as to whether it might be possible for $\sphericalangle X = \sphericalangle ABZ$ plus $\sphericalangle ACY$ while $\sphericalangle X' = \sphericalangle ABZ'$ minus $\sphericalangle ACY'$, in which case similarity of triangles XYZ and $X'Y'Z'$ would not exist. We shall answer by saying that it is possible to demonstrate that the signs must be alike when corresponding angle values are considered in triangles XYZ and $X'Y'Z'$. That is, if $\sphericalangle X$ is the sum of two angles, then $\sphericalangle X'$ must be equal to the sum of two angles. If $\sphericalangle X$ is equal to the difference of two angles, then $\sphericalangle X'$ must be equal to the difference of two angles. Similarly for the other angle pairs of triangles XYZ and $X'Y'Z'$. We again omit the rather extended proof.

There is a very important corollary of theorem 12 which we propose to record as a theorem itself. Since a straight line may be regarded as the limiting case of a circle, we have

Theorem 13. Circles BPC, CPA, and APB are met by a straight line through P in points X, Y, and Z respectively. Points X, Y, and Z are reflected with respect to BC, CA, and AB obtaining points X', Y', and Z'. Points X', Y', and Z' lie on a straight line through Q, the reflective partner of point P, and $(XY/X'Y') = (YZ/Y'Z') = (ZX/Z'X')$.

As a special case of theorem 13, we may again suppose that P is the orthocenter of triangle ABC. Circles BPC, CPA, and APB are then equal and when reflected about the sides of triangle ABC each becomes the circumcircle of triangle ABC. If a straight line through the orthocenter meets circles BPC, CPA, and APB in points X, Y, and Z we find that when these points are reflected with respect to BC, CA, and AB then points X', Y', and Z' become a single point on the circumcircle of triangle ABC. If we call this point X', we easily see that the midpoints of $X'X$, $X'Y$, and $X'Z$ are collinear and the line through these points is the Simson line of X' with respect to triangle ABC. Approaching from this direction we may determine many of the properties of Simson lines.

Having shown that triangle XYZ is reflective with respect to triangle ABC (theorem 12), we wonder whether triangle ABC could be reflective with respect to triangle XYZ. A little thought shows that for this to be true circles XCY, YAZ, and ZBX would have to intersect at a common point lying on circle ABC. Is it possible for this to occur? Let us return to Fig. 3 used in preparing the proof of theorem 2 and reverse the inversion carried out there. Using k, the same constant of inversion, and inverting with P as center of inversion, we list these results. Circles BPC, CPA, and APB invert into lines $B''C''$, $C''A''$, and $A''B''$ while the fourth circle $PXYZ$ inverts into line $X''Y''Z''$ all of which are shown in Fig. 3.

For circles XCY, YAZ, and ZBX to meet at a point on circle ABC, it would be necessary in the inverted figure to have circles $X''C''Y''$, $Y''A''Z''$, and $Z''B''X''$ meet on circle $A''B''C''$. This is exactly what happens because the four circumcircles of the four triangles formed by four lines in a plane have a point in common. The proof of this theorem may be found in geometrical texts and will not be given here. So the circles XCY, YAZ, and ZBX have a point K in common which lies on circle ABC. Using theorem 12 we may now reflect A with respect to YZ, B with respect to ZX, and C with respect to XY obtaining triangle $A'B'C'$ which will be inversely similar to triangle ABC. Triangles XYZ and ABC are then mutually reflective and we add two other theorems to our collection.

Theorem 14. Circles BPC, CPA, and APB are met by a fourth circle through P in points X, Y, and Z respectively. Circles XCY, YAZ, and ZBX have point K in common which lies on circle ABC.

Theorem 15. Circles BPC, CPA, and APB are met by a fourth circle through P in points X, Y, and Z respectively. A is reflected with respect to YZ, B is reflected with respect to ZX, and C is reflected with respect to XY. Triangle $A'B'C'$ thus determined is inversely similar to triangle ABC.

We have now shown that if triangle XYZ is reflective with respect to triangle ABC, then triangle ABC is reflective with respect to triangle XYZ. We may extend this result. In Fig. 1 we have four circles through point P. Thus far we have considered circles BPC, CPA, and APB as the three original circles with XYZ the fourth circle through P. It is evident that we may use any three of these circles as the original circles through P and the other as a fourth circle through this point. We list the four possibilities of which the first has already been discussed and listed as theorems 12 and 15.

(1) Three circles have point P in common and their other points of intersection form triangle ABC. A fourth circle through P meets circles BPC, CPA, and APB in points X, Y, and A respectively. Triangles XYZ and ABC are mutually reflective.

(2) Three circles have point P in common and their other points of intersection form triangle AYZ. A fourth circle through P meets circles YPZ, ZPA, and APY in points X, B, and C respectively. Triangles XBC and AYZ are mutually reflective.

(3) Three circles have point P in common and their other points of intersection form triangle BZX. A fourth circle through P meets circles ZPX, XPB, and BPZ in points Y, C, and A respectively. Triangles YCA and BZX are mutually reflective.

(4) Three circles have point P in common and their other points of intersection form triangle CXY. A fourth circle through P meets circles XPY, YPC, and CPX in points Z, A, and B respectively. Triangles ZAB and CXY are mutually reflective.

Remembering the notation for (2), (3), and (4) will involve some difficulty unless a simple device of some sort is employed as an aid to

memory. For this purpose let us arrange the letters X, Y, and Z determined as indicated in (1) in one horizontal row and the letters A, B, and C of (1) in a second horizontal row directly beneath them (Fig. 10). The six letters then form two horizontal rows and three vertical columns. The first column contains the letters X and A, the second column the letters Y and B, and the third column the letters Z and C. We shall call X and A of column one corresponding vertices. Likewise, Y and B will form a corresponding pair, as will Z and C. B and C will be called non-corresponding vertices with respect to X, X and Z will be non-corresponding vertices with respect to B, while A and B will be non-corresponding vertices with respect to Z, etc.

Proceeding from left to right (Fig. 10), without regard to row, we now select one letter from each of the three columns thereby determining the vertices of a triangle. The three remaining letters, reading from left to right, will determine a second triangle and the two triangles thus determined will be mutually reflective. This scheme is seen to give the four following mutually reflective pairs of

X	Y	Z
A	B	C

Figure 10

triangles which are seen to correspond to (1), (4), (3), and (2) just discussed. They are

$$XYZ \quad XYC \quad XBZ \quad XBC$$
$$ABC \quad ABZ \quad AYC \quad AYZ$$

After a pair of mutually reflective triangles are thus determined, how shall we conduct our reflections so that the inversely similar triangles may be obtained? Consider, for example, triangles XBZ and AYC, the third reflective pair listed above. Let us list the two triangles in the form $\dfrac{XBZ}{AYC}$ just shown. Then any vertex of one is reflected with respect to the side formed by the non-corresponding vertices of the other triangle. Thus, for these two triangles, X reflects with respect to YC, B reflects with respect to AC, and Z reflects with respect to AY giving triangle $X'B'Z'$ which is inversely similar to triangle XBZ. On the other hand A reflects with respect to BZ, Y reflects with respect to XZ, and C reflects with respect to XB giving triangle $A'Y'C'$ which is inversely similar to triangle AYC.

In this manner we may easily remember the four sets of triangles that are mutually reflective and also the manner in which the vertices of one triangle are to be reflected with respect to the sides of its reflective partner.

Suppose the four circles of Fig. 1 are inverted with respect to some point in the plane other than P. Circles BPC, CPA, and APB invert into three circles through P'' with their other points of intersection forming

triangle $A''B''C''$. Circle $PXYZ$ will invert into a fourth circle through P'' meeting circles $B''P''C''$, $C''P''A''$, and $A''P''B''$ at points X'', Y'', and Z'' respectively. Then, from theorems 12 and 15, triangles $X''Y''Z''$ and $A''B''C''$ are mutually reflective. Hence

Theorem 16. A pair of mutually reflective triangles invert into another pair of mutually reflective triangles.

Theorem 17. The six points X, Y, Z and A, B, C determining four sets of mutually reflective triangles invert into six points X'', Y'', Z'' and A'', B'', C'' that determine another set of four mutually reflective triangles.

With these theorems we must bring our discussion to a conclusion. Many other fascinating facts pertaining to reflectivity have been uncovered by the author during the past thirty years while searching the field of modern geometry. It would be very interesting to record the author's method of locating the self-homologous point of inversely similar triangles XYZ and $X'Y'Z'$ as well as that of triangles ABC and $A'B'C'$. Also, it is quite surprising to find that if P and Q be a pair of reflective points with respect to triangle ABC, they are also reflective points with respect to triangles BHC, CHA, and AHB where H is the orthocenter of triangle ABC. However, these and other matters must await discussion at some future time. For the present we are fearful lest this effort has been of undue length.

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Qualifying exam.

$$f(z) = u + iv$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

I can't remember either regular or king size functions
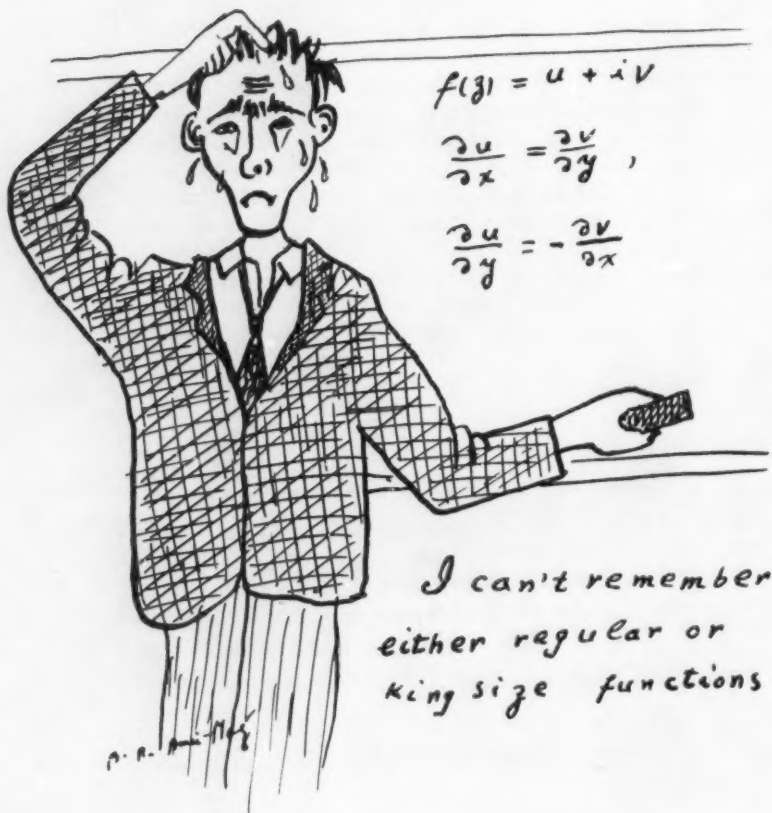
SPRAYS AND CAUCHY'S DISTRIBUTION

C. A. Wilkins

If particles are projected from the origin equally in all directions from $\theta = 0$ to $\theta = -\pi$, then the density of hits on the line $y = -p$ has the form

$$f(x) = \frac{1}{\pi} \frac{p}{p^2 + x^2}$$

i. e. Cauchy's distribution with parameter p[1]. The following alternative method for generating this distribution was noticed during work on a problem connected with spraying.

Consider, instead of a point source of particles, a circular source $x^2 + y^2 = r^2$. The particles are assumed to be uniformly distributed around its circumference and are projected towards $y = -p$ tangentially along the lines

$$x \cos \theta + y \sin \theta = r \tag{a}$$

θ then obeys the uniform distribution on $(-\pi, \pi)$.[2] If the tangent corresponding to θ intersects $y = -p$ at $(x, -p)$ then from (a)

$$\theta = \cot^{-1} \frac{x}{p} \pm \cos^{-1} \frac{r}{\sqrt{x^2 + p^2}}$$

Hence the distribution of x, i.e. the density of hits on $y = -p$, is given by[3]

$$f(x) = \frac{1}{2\pi} \left\{ \left| \frac{p}{x^2 + p^2} + \frac{rx}{(x^2 + p^2)\sqrt{x^2 + p^2 - r^2}} \right| + \left| \frac{p}{x^2 + p^2} - \frac{rx}{(x^2 + p^2)\sqrt{x^2 + p^2 - r^2}} \right| \right\}$$

If $p \geqq r$, we may drop the modulus signs, and $f(x)$ reduces to

$$f(x) = \frac{1}{\pi} \frac{p}{x^2 + p^2}$$

Thus exactly the same distribution results as if the particles had been projected from a point source at the origin. This has an application in spraying. Consider two sprays of the common revolving nozzle type, mounted so that the centres of revolution may be considered coincident, with the nozzles revolving in opposite directions and emitting thin jets of the spraying fluid horizontally perpendicular to the revolving arms (Figure 1). The angular velocities of the arms are equal, and the amounts of fluid ejected per second.

Take the origin at the common centre of revolution, and axes parallel

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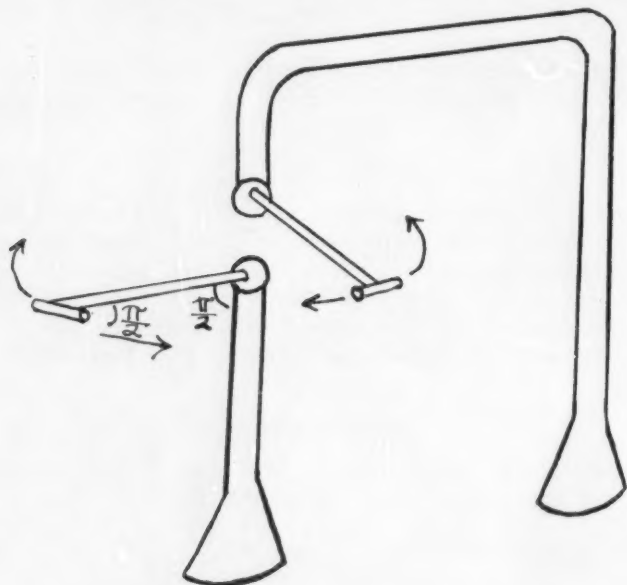and perpendicular to a line of thickly growing plants, AB. (Figure 2)

Figure 1

    Diagram to illustrate the spraying apparatus described in the text. For our purpose, the separate members are identical except that the arms revolve in opposite directions.
    The straight arrows represent the directions of the jets of spray; the curved arrows represent the angular velocities.

Suppose that the plants intercept all the fluid sprayed in directions between $\theta = -\theta_1$, and $\theta = -\pi + \theta_1$. Let p be the distance from the spraying apparatus to AB. (Figure 2)

A little reflection will show that, if K is the total amount of spray ejected from the apparatus in a given time, then the amount deposited between $x = a$ and $x = b$ is equal to

$$\frac{1}{2} K \frac{1}{\pi} \int_a^b \frac{p}{p^2 + x^2} dx , \qquad (-p \cot \theta_1 \leqq a \leqq b \leqq p \cot \theta_1) .$$

If $p < r$, the following distribution results:

$$f(x) = \frac{1}{\pi} - \frac{r|x|}{(x^2 + p^2)\sqrt{x^2 + p^2 - r^2}}$$

A sketch of this function is given in Figure 3. This distribution has no median[4] in the strict sense, discontinuities at $x = \pm\sqrt{r^2 - p^2}$, no

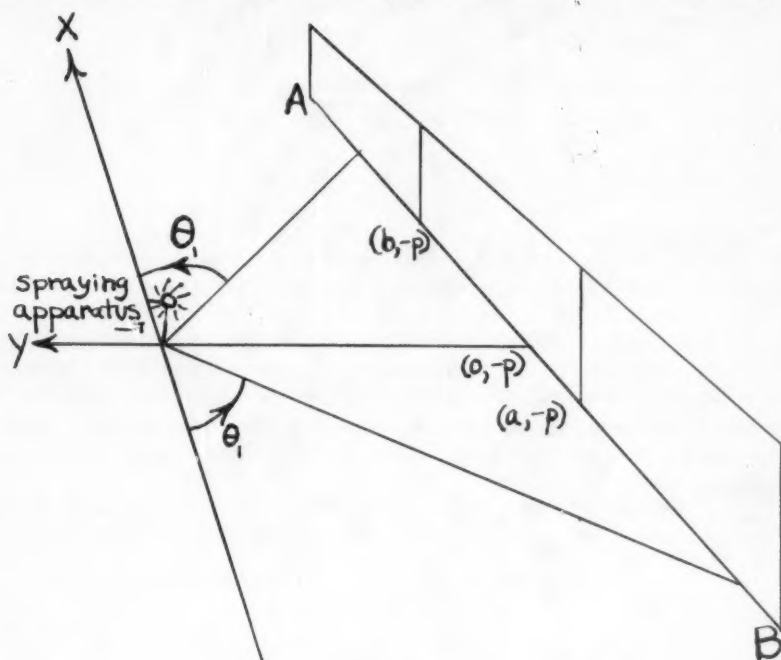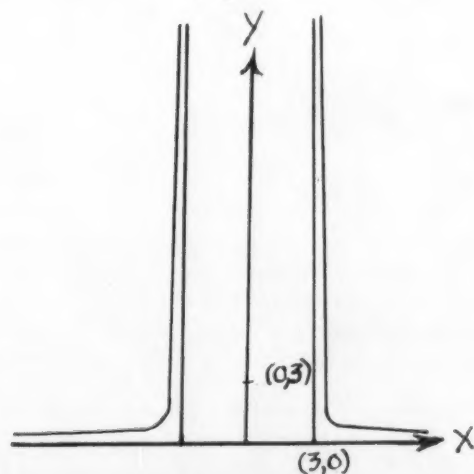stationary modes[5], infinite or indeterminate moments[6], and a range[7]

Figure 2

Figure 3
Graph to show the behavior of $f(x)$ when $r = 5$, $p = 4$.

given by $x^2 \geqq r^2 - p^2$. Its cumulative function[8] follows:

$$F(x) = \frac{1}{\pi} \sin^{-1}\left(\frac{r}{\sqrt{x^2+p^2}}\right), \qquad x \leqq -\sqrt{r^2-p^2},$$

$$= \frac{1}{2}, \qquad\qquad -\sqrt{r^2-p^2} < x \leqq \sqrt{r^2-p^2},$$

$$= 1 - \frac{1}{\pi}\sin^{-1}\left(\frac{r}{\sqrt{x^2+p^2}}\right), \quad \sqrt{r^2-p^2} < x.$$

The author has used this distribution as a class example to demonstrate that not all naturally occurring distributions have means, modes, or unique medians.

Both methods of generating Cauchy's distribution lead to graphic demonstrations that the quartiles[9] are located at $x = \pm p$. With the model of this note, take the case $p = r$, and construct the tangents to the circle $x^2 + y^2 = p^2$, that are perpendicular to the x-axis. Then half the total amount of particles is trapped symmetrically between these lines, so that the quartiles must be at $x = \pm p$.

Such demonstrations give the student a good intuitive idea of the nature of percentiles, which otherwise may remain abstract definitions.

REFERENCES

1. J. V. Uspensky, INTRODUCTION TO MATHEMATICAL PROBABILITY, pp. 242-243, McGraw-Hill, New York, 1939.

2. A. M. Mood, INTRODUCTION TO THE THEORY OF STATISTICS, pp. 107-108, McGraw-Hill, New York, 1950.

3. Ibid., pp. 193-196.

4. Ibid., p. 94.

5. Ibid., p. 55, p. 94.

6. Ibid., pp. 93-94.

7. Ibid., p. 51.

8. Ibid., pp. 76-82.

9. C. E. Weatherburn, A FIRST COURSE IN MATHEMATICAL STATISTICS, p. 3, Cambridge University Press, 1952.

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IDEALS OF SQUARE SUMMABLE POWER SERIES

James Rovnyak[1]

Let $G(z)$ be the Hilbert space of square summable power series with $\langle \Sigma a_n z^n, \Sigma b_n z^n \rangle = \Sigma a_n \overline{b}_n$. By an ideal of $G(z)$ we mean a vector subspace M of $G(z)$ such that $zf(z)$ belongs to M whenever $f(z)$ belongs to M. An ideal M of $G(z)$ will be called closed if it is a closed subset of $G(z)$ for the norm metric. In addition to the fundamental theorems on Hilbert space and orthogonality given by Halmos [2] chapter 1, we will take for granted the fact that the set of all formal power series with complex coefficients forms an integral domain with the ordinary definitions for multiplication and addition.

Theorems 1-7 which follow may be considered as lemmas to Theorems 8 and 9 in which the main results of this paper are contained.

THEOREM 1: If M is a non-zero closed ideal of $G(z)$, then the set of elements $zf(z)$ where $f(z)$ ranges in M is a closed vector subspace of M which is properly contained in M.

Proof: Let N be the set of elements $zf(z)$ where $f(z)$ ranges in M. If $g_1(z) = zf_1(z)$ and $g_2(z) = zf_2(z)$ are in N, then for complex numbers α and β, $\alpha g_1(z) + \beta g_2(z) = z[\alpha f_1(z) + \beta f_2(z)]$, where, since M is a vector subspace of $G(z)$, $\alpha f_1(z) + \beta f_2(z)$ is in M, and $\alpha g_1(z) + \beta g_2(z)$ is in N.

Let $g_n(z) = zf_n(z)$ be a Cauchy sequence in N. Since multiplication by z is a linear isometric transformation, $||f_m(z) - f_n(z)|| = ||g_m(z) - g_n(z)||$, and $f_n(z)$ is a Cauchy sequence in M. Let $f(z)$ be the limit of $f_n(z)$. Since M is closed by hypothesis, $f(z) = \lim_n f_n(z)$ is in M, and $zf(z)$ is in N. From $||g_n(z) - zf(z)|| = ||f_n(z) - f(z)||$, it follows that $\lim_n g_n(z) = zf(z)$, and that N is a closed vector subspace of M.

Since M is an ideal, $N \subset M$. To see that the inclusion is proper, argue by contradiction: suppose that $M \subset N$. Then if $g(z) = \Sigma a_n z^n$ is in M, $g(z) = zf(z)$ for some $f(z)$ in M, and $g(z)/z$ is defined and belongs to M, implying that $a_0 = 0$. Continuing the argument inductively, for every $m = 1, 2, 3, \cdots$, $g(z)/z^m$ is defined and belongs to M, and for every $n = 0, 1, 2, \cdots$, $a_n = 0$, contradicting the assumption that M is not the zero subspace.

THEOREM 2: Let M be a non-zero closed ideal of $G(z)$, and let $B(z)$ be an element of M of norm 1 orthogonal to every series $zf(z)$ where $f(z)$ ranges in M. Then $B(z)$, $B(z)z$, $B(z)z^2$, $\cdots$ is an orthonormal set in $G(z)$.

In the light of Theorem 1, the existence of such a $B(z)$ is guaranteed

by a theorem on orthogonality. (Halmos [2] p. 23.)

Proof: For $m = 1, 2, 3, \cdots, ||B(z)z^m|| = ||B(z)|| = 1$. If m and n are integers such that $m \neq n$,

$$\langle B(z)z^m, B(z)z^n\rangle = \langle B(z)z^{m-n}, B(z)\rangle \quad \text{if } m > n$$

$$= \langle B(z), B(z)z^{n-m}\rangle \quad \text{if } m < n.$$

But in either case, $\langle B(z)z^m, B(z)z^n\rangle = 0$ since $B(z)z^k$ $(k = 1, 2, 3, \cdots)$ is in the set of elements $zf(z)$ where $f(z)$ ranges in **M**.

THEOREM 3: If $B(z)$ is a square summable power series such that $B(z)$, $B(z)z$, $B(z)z^2$, $\cdots$ is an orthonormal set in **G**(z), then for $f(z)$ in **G**(z), the formal power series product $B(z)f(z)$ is in **G**(z) and $||B(z)f(z)|| = ||f(z)||$.

Proof: Suppose that $f(z) = \Sigma a_n z^n$. Then $B(z)f(z) = \Sigma a_n B(z)z^n$ [2] converges in the metric of **G**(z) and $||B(z)f(z)||^2 = ||f(z)||^2$ by what Halmos refers to as the generalized Pythagorean theorem, [2] p. 19.

THEOREM 4: If $B(z)$ is a square summable power series such that $B(z)$, $B(z)z, B(z)z^2, \cdots$ is an orthonormal set in **G**(z), then for each complex number w with $|w| < 1$, $K(w, z) = [1 - \overline{B}(w)B(z)]/(1 - \overline{w}z)$ is a square summable power series in z. If $|w_1| < 1$ and $|w_2| < 1$, $\langle K(w_1, z), K(w_2, z)\rangle = K(w_1, w_2)$. For $|w| < 1$, $|B(w)| \leq 1$ with equality holding only when $B(z) = c$ where c is a constant of absolute value 1.

Proof: It is clear that for $|w| < 1$, $(1 - \overline{w}z)^{-1} = \Sigma \overline{w}^n z^n$ is in **G**(z) since $\Sigma|w|^{2n} = (1 - |w|^2)^{-1}$ is a convergent geometric series, and that if $f(z)$ is any square summable power series,

(1)
$$f(w) = \langle f(z), (1 - \overline{w}z)^{-1}\rangle.$$

Since $(1 - \overline{w}z)^{-1}$ is in **G**(z), by Theorem 3, $B(z)(1 - \overline{w}z)^{-1}$ is in **G**(z), and hence, $K(w, z)$ is in **G**(z).

By Theorem 3, $f(z) \to B(z)f(z)$ defines a linear isometric transformation of **G**(z) into itself. An immediate c o n s e q u e n c e of the relation $\langle f, g\rangle = \frac{1}{4}[||f+g||^2 - ||f-g||^2 + i||f+ig||^2 - i||f-ig||^2]$ is that such a transformation preserves inner products. This fact, in addition to (1), justifies the following computation.

$$\langle K(w_1, z), K(w_2, z)\rangle = \langle(1 - \overline{w}_1 z)^{-1}, (1 - \overline{w}_2 z)^{-1}\rangle$$

$$-\langle(1 - \overline{w}_1 z)^{-1}, \overline{B}(w_2)B(z)(1 - \overline{w}_2 z)^{-1}\rangle$$

$$-\langle\overline{B}(w_1)B(z)(1 - \overline{w}_1 z)^{-1}, (1 - \overline{w}_2 z)^{-1}\rangle$$

$$+\langle\overline{B}(w_1)B(z)(1 - \overline{w}_1 z)^{-1}, \overline{B}(w_2)B(z)(1 - \overline{w}_2 z)^{-1}\rangle$$

---

2. This infinite sum may be regarded either as a formal process or as a limit in the Hilbert space metric. When the limit exists it is identical with the formal sum.

$$= \frac{1}{1-\overline{w}_1 w_2} - \frac{B(w_2)\overline{B}(w_1)}{1-\overline{w}_1 w_2} - \frac{\overline{B}(w_1)B(w_2)}{1-\overline{w}_1 w_2} + \frac{\overline{B}(w_1)B(w_2)}{1-\overline{w}_1 w_2}$$

$$= \frac{1-\overline{B}(w_1)B(w_2)}{1-\overline{w}_1 w_2} = K(w_1, w_2) \, .$$

Here we see that for $|w| < 1$,

$$\frac{1-|B(w)|^2}{1-|w|^2} = K(w, w) = ||K(w, z)||^2 \geqq 0$$

so that $|B(w)| \leq 1$. This inequality is strict unless $K(w, z) = [1-\overline{B}(w)B(z)]/(1-\overline{w}z) = 0$ identically. Therefore, in this case $B(z)$ is a constant and this formula evaluates its magnitude.

THEOREM 5 : If $B(z)$ is a square summable power series such that $B(z)$, $B(z)z$, $B(z)z^2$, $\cdots$ is an orthonormal set in $\mathbf{G}(z)$, then a necessary and sufficient condition that an element $g(z)$ of $\mathbf{G}(z)$ be of the form $g(z) = B(z)f(z)$ for some $f(z)$ in $\mathbf{G}(z)$ is that $<z^m g(z), B(z)> = 0$ for every $m = 1, 2, 3, \cdots$. In this case, $f(z) = \Sigma a_n z^n$ where $a_n = <g(z), B(z)z^n> \ (n = 0, 1, 2, \cdots)$.

This theorem has the interesting interpretation of giving a necessary and sufficient condition for a square summable power series having a representation with the particular orthonormal set $B(z)$, $B(z)z$, $B(z)z^2$, $\cdots$.

Proof of necessity : If $g(z) = B(z)f(z)$ for some $f(z)$ in $\mathbf{G}(z)$, then whenever $m = 1, 2, 3, \cdots$,

$$<z^m g(z), B(z)> = <\Sigma a_n B(z)z^{m+n}, B(z)>$$

$$= \Sigma a_n <B(z)z^{m+n}, B(z)> = \Sigma 0 = 0 \, .$$

In this case, taking the inner product of both sides of the equation $g(z) = \Sigma a_n B(z)z^n$ with $B(z)z^k$, we see that $a_k = <g(z), B(z)z^k> \ (k = 0, 1, 2, \cdots)$.

Proof of sufficiency : If, on the other hand, the condition holds for $g(z)$ in $\mathbf{G}(z)$, set $f(z) = \Sigma a_n z^n$ where $a_n = <g(z), B(z)z^n> \ (n = 0, 1, 2, \cdots)$. This series is in $\mathbf{G}(z)$ since by Bessel's inequality,

$$\Sigma |<g(z), B(z)z^n>|^2 \leqq ||g(z)||^2 < \infty \, .$$

Now, observing that for $|w| < 1$,

(2) $$\frac{\overline{w}z\,B(z)}{1-\overline{w}z} = \frac{B(z)}{1-\overline{w}z} - B(z) \, ,$$

(3) $$f(w) = \Sigma a_n s^n = \Sigma <g(z), B(z)z^n>w^n$$

$$= <g(z), B(z)\Sigma \overline{w}^n z^n> = <g(z), B(z)(1-\overline{w}z)^{-1}>$$

by our construction,

$$wf(w) = w\langle g(z), B(z)(1-\overline{w}z)^{-1}\rangle = \langle zg(z), \overline{w}zB(z)(1-\overline{w}z)^{-1}\rangle$$

$$= \langle zg(z), B(z)(1-\overline{w}z)^{-1}\rangle - \langle zg(z), B(z)\rangle$$

$$= \langle zg(z), B(z)(1-\overline{w}z)^{-1}\rangle$$

since $zg(z)$ is orthogonal to $B(z)$ by hypothesis.

Arguing by mathematical induction, we conclude that for every $n = 0, 1, 2, \cdots$,

(4) $$w^n f(w) = \langle z^n g(z), B(z)(1-\overline{w}z)^{-1}\rangle .$$

If $B(z) = \Sigma B_n z^n$, using (4), we obtain by Theorem 3,

(5) $$B(w)f(w) = \Sigma B_n f(w)w^n = \Sigma B_n \langle z^n g(z), B(z)(1-\overline{w}z)^{-1}\rangle$$

$$= \langle \Sigma B_n g(z)z^n, B(z)(1-\overline{w}z)^{-1}\rangle$$

$$= \langle B(z)g(z), B(z)(1-\overline{w}z)^{-1}\rangle$$

$$= \langle g(z), (1-\overline{w}z)^{-1}\rangle = g(w) .$$

Since w is arbitrary, $g(z) = B(z)f(z)$.

THEOREM 6: Let $B(z)$ be a square summable power series such that $B(z)$, $B(z)z$, $B(z)z^2$, $\cdots$ is an orthonormal set in $\mathbf{G}(z)$, let $f(z)$ be an element of $\mathbf{G}(z)$, and let $g(z) = B(z)f(z)$. Then a necessary and sufficient condition that $f(z)$, $f(z)z$, $f(z)z^2$, $\cdots$ be an orthonormal set in $\mathbf{G}(z)$ is that $g(z)$, $g(z)z$, $g(z)z^2$, $\cdots$ be an orthonormal set in $\mathbf{G}(z)$.

Proof: This theorem follows from Theorem 3 and the fact that a linear isometric transformation preserves inner products.

THEOREM 7: If $B_1(z)$ and $B_2(z)$ are square summable power series such that $B_1(z)$, $B_1(z)z$, $B_1(z)z^2$, $\cdots$ and $B_2(z)$, $B_2(z)z$, $B_2(z)z^2$, $\cdots$ are orthonormal sets in $\mathbf{G}(z)$, then a necessary and sufficient condition that $B_2(z) = cB_1(z)$ where c is a constant of absolute value 1 is that $\langle B_1(z)z^m, B_2(z)\rangle = \langle B_2(z)z^m, B_1(z)\rangle = 0$ for every $m = 1, 2, 3, \cdots$.

Proof of necessity: Suppose that $B_2(z) = cB_1(z)$. Then whenever $m = 1, 2, 3, \cdots$, $\langle B_1(z)z^m, B_2(z)\rangle = \overline{c}\langle B_1(z)z^m, B_1(z)\rangle = 0$ and $\langle B_2(z)z^m, B_1(z)\rangle = c\langle B_1(z)z^m, B_1(z)\rangle = 0$.

Proof of sufficiency: By Theorem 5, there exist square summable power series $B_{12}(z)$ and $B_{21}(z)$ such that $B_1(z) = B_2(z)B_{21}(z)$ and $B_2(z) = B_1(z)B_{12}(z)$. By Theorem 6, $B_{12}(z)$, $B_{12}(z)z$, $B_{12}(z)z^2$, $\cdots$ and $B_{21}(z)$, $B_{21}(z)z$, $B_{21}(z)z^2$, are orthonormal sets in $\mathbf{G}(z)$.

Since by hypothesis $B_1(z)$ has norm 1, $B_1(z) \neq 0$. Hence it follows from

$$B_1(z)[1 - B_{12}(z)B_{21}(z)] = B_1(z) - (B_1(z)B_{12}(z))B_{21}(z)$$

$$= B_1(z) - B_2(z)B_{21}(z) = B_1(z) - B_1(z) = 0$$

that $B_{12}(z)B_{21}(z) = 1$, and that for $|w| < 1$,

(6) $$|B_{12}(w)||B_{21}(w)| = 1 .$$

Both $B_{12}(z)$ and $B_{21}(z)$ satisfy the hypothesis to Theorem 4. Hence, both $|B_{12}(w)| \leq 1$ and $|B_{21}(w)| \leq 1$, hence, both $|B_{12}(w)| = 1$ and $|B_{21}(w)| = 1$, since an assumption to the contrary contradicts (6). Now, by the final statement of Theorem 4, $B_{12}(z)$ and $B_{21}(z)$ are constants of absolute value 1, and the theorem follows.

We are now in a position to characterize the non-zero closed ideals of $\mathbf{G}(z)$.

THEOREM 8 : Let $B(z)$ be a square summable power series such that $B(z)$, $B(z)z$, $B(z)z^2$, $\cdots$ is an orthonormal set in $\mathbf{G}(z)$, and let $\mathbf{M}(B)$ be the set of elements $g(z) = B(z)f(z)$ where $f(z)$ ranges in $\mathbf{G}(z)$. Then $\mathbf{M}(B)$ is a non-zero closed ideal of $\mathbf{G}(z)$. Furthermore, if $\mathbf{M}$ is any non-zero closed ideal of $\mathbf{G}(z)$, then there is a square summable power series $B(z)$ such that $B(z)$, $B(z)z$, $B(z)z^2$, $\cdots$ is an orthonormal set in $\mathbf{G}(z)$ and such that $\mathbf{M} = \mathbf{M}(B)$. This $B(z)$ is uniquely determined within a constant factor of absolute value 1.

Proof: By Theorem 3, $\mathbf{M}(B) \subset \mathbf{G}(z)$. If $g_1(z) = B(z)f_1(z)$ and $g_2(z) = B(z)f_2(z)$ are in $\mathbf{M}(B)$, then for complex numbers $\propto$ and β, $\propto g_1(z) + \beta g_2(z) = B(z)[\propto f_1(z) + \beta f_2(z)]$ and $\propto g_1(z) + \beta g_2(z)$ is in $\mathbf{M}(B)$.

Let $g_n(z) = B(z)f_n(z)$ be a Cauchy sequence in $\mathbf{M}(B)$. By Theorem 3, $||f_m(z) - f_n(z)|| = ||g_m(z) - g_n(z)||$, and $f_n(z)$ is a Cauchy sequence in $\mathbf{G}(z)$. Let $f(z)$ be the limit of $f_n(z)$. Since $||g_n(z) - B(z)f(z)|| = ||f_n(z) - f(z)||$, it follows that $\lim_n g_n(z) = B(z)f(z)$, where, by its form, $B(z)f(z)$ is in $\mathbf{M}(B)$. Thus $\mathbf{M}(B)$ is a closed vector subspace of $\mathbf{G}(z)$ which is obviously not the zero subspace.

It is clear that $\mathbf{M}(B)$ is an ideal, since if $g(z) = B(z)f(z)$ is in $\mathbf{M}(B)$, $zg(z) = B(z)(zf(z))$ is in $\mathbf{M}(B)$.

To complete the proof, let $\mathbf{M}$ be any non-zero closed ideal of $\mathbf{G}(z)$, and let $B(z)$ be an element of $\mathbf{M}$ of norm 1 orthogonal to every series $zf(z)$ where $f(z)$ ranges in $\mathbf{M}$ (see the comment after Theorem 2). By Theorem 2, $B(z)$, $B(z)z$, $B(z)z^2$, $\cdots$ is an orthonormal set in $\mathbf{G}(z)$.

If $g(z)$ is in $\mathbf{M}$, then for $m = 1, 2, 3, \cdots$, $\langle z^m g(z), B(z)\rangle = 0$ by the way in which we chose $B(z)$. By Theorem 5 then, there is an element $f(z)$ of $\mathbf{G}(z)$ such that $g(z) = B(z)f(z)$. Thus, $g(z)$ is in $\mathbf{M}(B)$, and $\mathbf{M} \subset \mathbf{M}(B)$.

If $g(z)$ is in $\mathbf{M}(B)$, then for some $f(z) = \Sigma a_n z^n$ in $\mathbf{G}(z)$, $g(z) = B(z)f(z)$. Set $f_k(z) = \sum_{n=0}^{k} a_n z^n$. Clearly, $\lim_k f_k(z) = f(z)$, and therefore, by Theorem 3, $\lim_k B(z)f_k(z) = B(z)f(z)$. Since $\mathbf{M}$ is a non-zero closed ideal, and since $B(z)$ is in $\mathbf{M}$, it follows by induction that for each $k = 1, 2, 3, \cdots$, $B(z)f_k(z)$ is in $\mathbf{M}$, and hence that $B(z)f(z)$ is in $\mathbf{M}$. We conclude that for this $B(z)$, $\mathbf{M} = \mathbf{M}(B)$.

Finally, if $\mathsf{M} = \mathsf{M}(B_1) = \mathsf{M}(B_2)$, we have in particular that $B_2(z)$ is in $\mathsf{M}(B_1)$. Hence, there is an element $f(z) = \Sigma a_n z^n$ of $\mathsf{G}(z)$ such that $B_2(z) = B_1(z)f(z)$. Also,

$$\langle B_2(z)z^m, B_1(z) \rangle = \langle \Sigma a_n B_1(z)z^{m+n}, B_1(z) \rangle = \Sigma a_n \langle B_1(z)z^{m+n}, B_1(z) \rangle = \Sigma 0 = 0$$

whenever $m = 1, 2, 3, \cdots$.

Similarly, $\langle B_1(z)z^m, B_2(z) \rangle = 0$ for $m = 1, 2, 3, \cdots$. The result now follows from Theorem 7.

We conclude with the following property of the non-zero closed ideals of $\mathsf{G}(z)$. Although the theorem is stated in terms of ideals $\mathsf{M}(B)$, the result can be reworded to include arbitrary non-zero closed ideals, M, by the previous theorem.

THEOREM 9: Let $B_1(z)$ and $B_2(z)$ be square summable power series such that $B_1(z)$, $B_1(z)z$, $B_1(z)z^2$, $\cdots$ and $B_2(z)$, $B_2(z)z$, $B_2(z)z^2$, $\cdots$ are orthonormal sets in $\mathsf{G}(z)$. Then $\mathsf{M}(B_1) \subset \mathsf{M}(B_2)$ if and only if there is a square summable power series $B_3(z)$ such that $B_1(z) = B_2(z)B_3(z)$. In this case, $B_3(z)$, $B_3(z)z$, $B_3(z)z^2$, $\cdots$ is an orthonormal set in $\mathsf{G}(z)$.

Proof: Suppose that $\mathsf{M}(B_1) \subset \mathsf{M}(B_2)$. Then in particular, $B_1(z)$ is in $\mathsf{M}(B_2)$, and, by definition, this means that there is a square summable power series $B_3(z)$ such that $B_1(z) = B_2(z)B_3(z)$. In this case, it follows from Theorem 6 that $B_3(z)$, $B_3(z)z$, $B_3(z)z^2$, $\cdots$ is an orthonormal set in $\mathsf{G}(z)$.

If, on the other hand, for some $B_3(z)$ in $\mathsf{G}(z)$, $B_1(z) = B_2(z)B_3(z)$, and if $g(z) = B_1(z)f(z)$ is in $\mathsf{M}(B_1)$, then $g(z) = (B_2(z)B_3(z))f(z) = B_2(z)(B_3(z)f(z)) = B_2(z)h(z)$, where, by Theorem 3, $h(z)$ is in $\mathsf{G}(z)$. Thus, $g(z)$ is in $\mathsf{M}(B_2)$, and $\mathsf{M}(B_1) \subset \mathsf{M}(B_2)$.

COROLLARY : Let $B_1(z)$ and $B_2(z)$ be square summable power series such that $B_1(z)$, $B_1(z)z$, $B_1(z)z^2$, $\cdots$ and $B_2(z)$, $B_2(z)z$, $B_2(z)z^2$, $\cdots$ are orthonormal sets in $\mathsf{G}(z)$. Then the following three conditions are equivalent.

(i) $\mathsf{M}(B_1) = \mathsf{M}(B_2)$

(ii) $\langle z^m B_1(z), B_2(z) \rangle = \langle z^m B_2(z), B_1(z) \rangle = 0$ $(m = 1, 2, 3, \cdots)$

(iii) $B_2(z) = cB_1(z)$ where c is a constant of absolute value 1.

Proof: Theorems 5, 7, and 9.

REFERENCES

1. P. R. Halmos, *Finite Dimensional Vector Spaces*, D. Van Nostrand Company, Inc., 1958.

2. P. R. Halmos, *Introduction to Hilbert Space*, Chelsea Publishing Company, 1951.

3. M. H. Stone, *Linear Transformations in Hilbert Space*, American Mathematical Society Colloquium Publications, vol. XV, New York, 1932.

4. Arne Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. 81, 17 pp. (1948).

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TEACHING OF MATHEMATICS

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SOME GEOMETRIC CONSIDERATIONS RELATED TO THE MEAN VALUE THEOREM

Roger Osborn

In the teaching of differential calculus, insufficient attention is given to the geometric interpretations of both the generating function for the mean value theorem (hereafter referred to as MVT) and the conclusion of the MVT. An article to this effect was written some 34 years ago by A. A. Bennett and published in the AMERICAN MATHEMATICAL MONTHLY ("The Consequences of Rolle's Theorem," Vol. 31, p. 40). The few words written here have little in common with those of Dr. Bennett except to emphasize the need for the understanding of the MVT which can come most easily from interpreting it geometrically.

Geometric interpretation of the MVT can be divided into three phases: (a) interpreting a generating function to be used in obtaining the desired results, (b) interpreting the fashion in which the generating function sat-isfies Rolle's Theorem, and (c) interpreting the conclusion of the MVT relative to the generating function.

Many generating functions (and by generating function is meant the function which will be shown to possess the properties of Rolle's Theo-rem and which will be differentiated to yield the conclusion of the MVT) are available. Various generating functions lead to various forms of the conclusion of the MVT. Let us suppose that the functions with which we deal all satisfy the ordinary continuity conditions imposed by the MVT.

To obtain the ordinary MVT, the generating functions may be inter-preted primarily as either area functions or length functions. Consider the function $f(x)$ and the chord joining $[a, f(a)]$ to $[b, f(b)]$. The generating func-tion which most texts ask the student to consider is

$$\phi(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right].$$

Some, but not nearly all, books interpret this correctly as the length of the

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vertical segment joining points on the curve and on the chord which have the same abcissa. The function may be constructed originally to have this meaning. Since this segment has zero length for $x = a$ or $x = b$, its length is a function satisfying the conditions and the conclusion of Rolle's Theorem. The results are interpreted as usual.

Another generating function which appears infrequently in textbooks, but which is by no means new, is the area of the triangle with vertices at A, B, X. (See Figure 1.) This area function may be given by
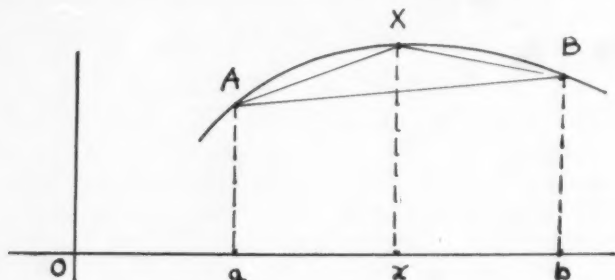
Figure 1

$$A(x) = \pm \frac{1}{2} \begin{vmatrix} a & f(a) & 1 \\ b & f(b) & 1 \\ x & f(x) & 1 \end{vmatrix}$$

in determinant form, or may be given in the equivalent notation

$$A(x) = \pm \frac{1}{2} \begin{bmatrix} a & b & x & a \\ f(a) & f(b) & f(x) & f(a) \end{bmatrix},$$

with this latter symbolism to be interpreted to mean (exclusive of coefficient)

$$af(b) + bf(x) + xf(a) - bf(a) - xf(b) - af(x),$$

a pattern not unlike the expansion of a determinant, but one which lends itself to the expression of the area of any polygon. It can be seen that this area reduces to zero when $x = a$ or $x = b$, and application of Rolle's Theorem generates the ordinary MVT. The latter result,

$$f(b) - f(a) - (b-a)f'(x_1) = 0$$

may be interpreted as a definition of a point x_1 at which the area of the triangle assumes a relative maximum or minimum value.

A third generating function could be the expression for the distance from X perpendicular to the chord AB. This distance assumes a relative maximum or minimum for the same x_1 as does the vertical distance discussed above.

Still another generating function may be taken to be the area subtended at the origin by the polygonal arc joining A, B, and X (see Figure 2). This function is

$$A(x) = \pm \frac{1}{2} \begin{bmatrix} 0 & a & x & b & 0 \\ 0 & f(a) & f(x) & f(b) & 0 \end{bmatrix}.$$

It may be observed that when $x = a$ and when $x = b$, $A(x)$ has the same value (the area of triangle OAB).
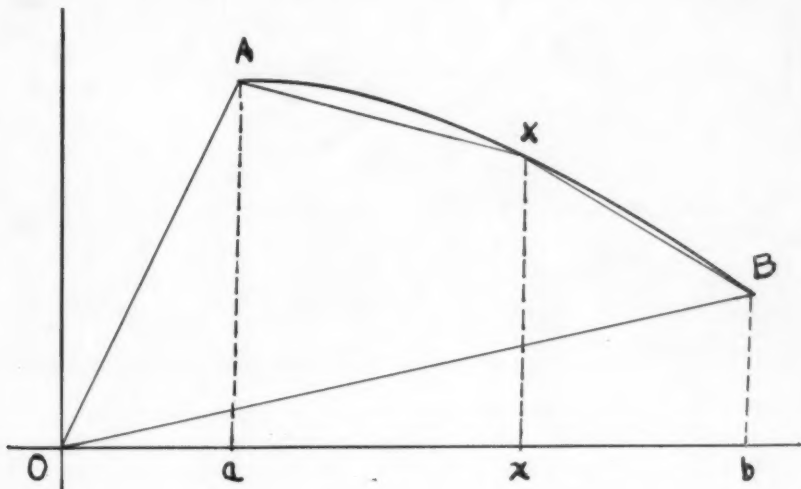
Figure 2

We differentiate this area function in a fashion similar to that used for differentiating determinants:

$$A'(x) = \pm \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & f(a) & f(x) & f(b) & 0 \end{bmatrix} \pm \frac{1}{2} \begin{bmatrix} 0 & a & x & b & 0 \\ 0 & 0 & f'(x) & 0 & 0 \end{bmatrix}$$

$$= \pm \frac{1}{2} [f(b) - f(a) - (b-a)f'(x)].$$

This result may be interpreted as the preceding one was — that since $A(a) = A(b)$, $A'(x)$ will be zero for some $a < x_1 < b$, or that $A(x)$ has a relative maximum or minimum for some x between a and b.

This latter generating function has the property that

$$\begin{bmatrix} 0 & a & x & b & 0 \\ 0 & f(a) & f(x) & f(b) & 0 \end{bmatrix}$$

is twice the area subtended at the origin by the polygonal arc. It is interesting to note that a similar quantity — twice the subtended area — occurs

in other considerations. It is the value of the parameter t in $\sinh t$, $\cosh t$, if the area is OIT in Figure 3. Similar comments apply to the argument of circular functions.

$(\cosh t, \sinh t)$
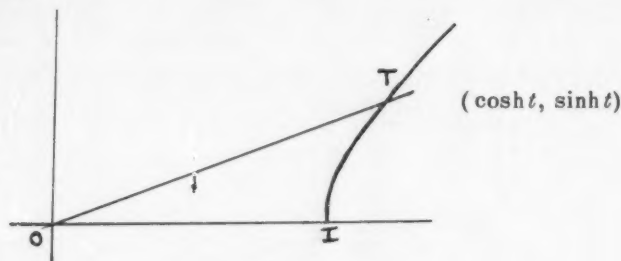
Figure 3

Finally, the generalized MVT may be obtained as a form of the ordinary MVT

$$\frac{f(b) - f(a)}{b - a} = f'(x_1), \quad a < x_1 < b .$$

Let $\begin{cases} x = F(t) \\ y = G(t) \end{cases}$ be the parameterization of $y = f(x)$ with suitable continuity restrictions. Then we conclude

$$\frac{G(t_2) - G(t_1)}{F(t_2) - F(t_1)} = \frac{G'(\overline{t})}{F'(\overline{t})}, \quad t_1 < \overline{t} < t_2 .$$

This form is not new, but is insufficiently emphasized. By taking this interpretation we are saved the need of presenting further geometric interpretations of a two function MVT.

The form of the generating function for the MVT gives rise to the final form of the MVT. For example, consider the polygonal area $AA'X'B'BXA$ of Figure 4.

$$A(x) = \pm \frac{1}{2} \begin{bmatrix} a & a & x & b & b & x & a \\ f(a) & F(a) & F(x) & F(b) & f(b) & f(x) & f(a) \end{bmatrix}$$

Now, $A(a) = A(b)$, and applying Rolle's Theorem, we have

$$A'(x) = \pm \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \cdots & F(a) & \cdots & F(b) & f(b) & \cdots & f(a) \end{bmatrix}$$

$$\pm \frac{1}{2} \begin{bmatrix} a & a & x & b & b & x & a \\ 0 & 0 & F'(x) & 0 & 0 & f'(x) & 0 \end{bmatrix}$$

$$= \pm \frac{1}{2} \{ [F(b) - F(a)] - [f(b) - f(a)] - (b-a)F'(x) + (b-a)f'(x) \} .$$

Since $A'(x) = 0$ for some $x = x_1$, we have
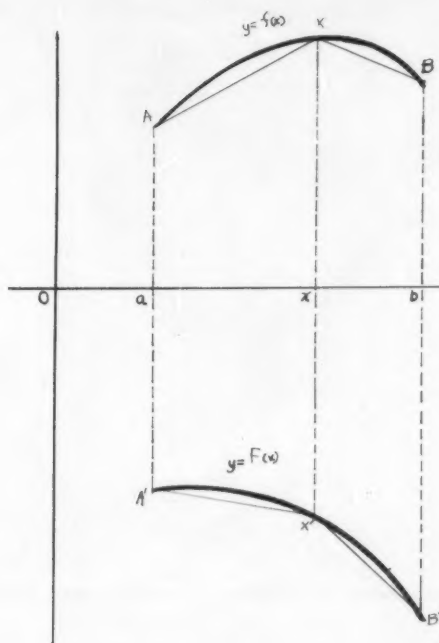
Figure 4

$$\frac{F(b)-F(a)}{b-a} - F'(x_1) = \frac{f(b)-f(a)}{b-a} - f'(x_1) .$$

This unusual form yields the ordinary MVT for the choice $F(x) = x$, and close inspection reveals that it is another form of the ordinary MVT for $G(x)$, $G(x)$ being defined to be $F(x) - f(x)$.

Note: parts of this paper were used in a paper given to the meeting of the Texas Section of the Mathematical Association of America on April 20, 1956.

The University of Texas

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + 2cy = 0.$$
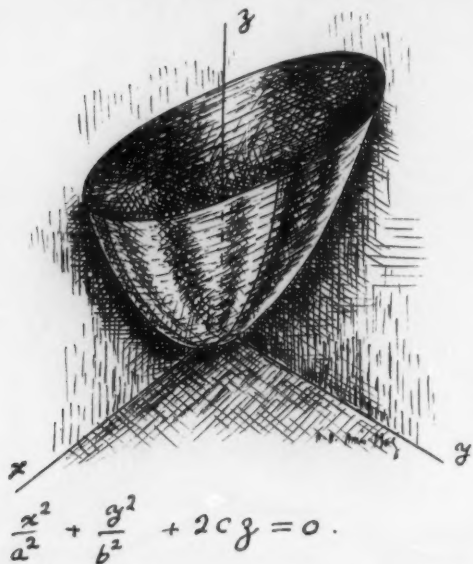
The idea expressed by Grassmann is essentially the one held at the present time; that is, a mathematical system called "geometry" is not necessarily a description of actual space. One must distinguish, of course, between the origin of a theory and the form to which it evolves. Geometry, like arithmetic, originated in things "practical," but to assert that any particular type of geometry is a description of physical space is to make a physical assertion, not a mathematical statement. In short, the modern viewpoint is that one must distinguish between mathematics and applications of mathematics.

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MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

TEST FOR THE RANK OF A MATRIX

Louis Brand

Consider a n by m matrix A of rank r. Shift the rows and columns of A to obtain a matrix B which has a non-singular r by r matrix P in the upper left corner. These operations do not alter the rank of a matrix; hence B has the same rank as A. Now partition B into four sub-matrices P, Q, R, S as shown:

(1)
$$B = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \text{ where } \det P \neq 0 .$$

If $n = r+k$, $m = r+j$, the sub-matrices P, Q, R, S are respectively r by r, r by j, k by r, k by j.

Since B is of rank r, the last k rows of B, namely (R, S), are linear combinations of the first r rows (P, Q). Hence there exist r constants c_{i1}, c_{i2}, $\cdots$, c_{ir} such that when the first r rows of B are multiplied by them and added, we get the $(r+i)$th row of B. Since $i = 1, 2, \cdots, k$, we thus obtain a k by r matrix C such that

$$R = CP , \quad S = CQ ;$$

since $C = RP^{-1}$ we have

(2)
$$S = RP^{-1}Q .$$

as a necessary condition that B is of rank r.

Conversely, if B can be partitioned as shown in (1), where P is a non-singular r by r matrix, the rank of B will be r if equation (2) holds. For let $C = RP^{-1}$; then

$$R = CP \quad \text{and} \quad S = CQ .$$

But this shows that the rows (R, S) are linear combinations of the rows (P, Q) and hence all determinants in B of order $> r$ must vanish and B is of rank r.

Example 1. The matrix

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$$A = \begin{pmatrix} 1 & 2 & | & 3 & 1 \\ \underline{2} & \underline{0} & | & \underline{2} & \underline{-2} \\ 3 & 1 & | & 4 & -2 \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

is of rank 2; for

$$P^{-1} = -\frac{1}{4}\begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix}, \quad RP^{-1} = \frac{1}{4}(2, 5), \quad RP^{-1}Q = \frac{1}{4}(16, -8) = S \ .$$

The matrix $C = (1/2, 5/4)$.

Example 2. The matrix

$$B = \begin{pmatrix} 1 & -3 & | & 0 & 0 \\ \underline{5} & \underline{-3} & | & \underline{3} & \underline{3} \\ 1 & 1 & | & 1 & 1 \\ 3 & -1 & | & 2 & 2 \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

is of rank 2; for

$$P^{-1} = \frac{1}{12}\begin{pmatrix} -3 & 3 \\ -5 & 1 \end{pmatrix}, \quad RP^{-1} = \frac{1}{3}\begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}, \quad RP^{-1}Q = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = S \ .$$

University of Houston

Mathematics according to David Hilbert (1862-), is a game played according to certain simple rules with meaningless marks on paper. This is rather a comedown from the architecture of the universe, but it is the final dry flower of a century of progress. The meaning of mathematics has nothing to do with the game, and mathematicians pass outside their proper domain when they attempt to give the marks meanings.

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A SIMPLE PROOF OF FERMAT'S LAST THEOREM
FOR $n = 6$ AND $n = 10$

Robert Breusch

Theorem 1. The system $x^2 + y^2 = u^2 + v^2$, $xy = 2uv$ has no solution in positive integers x, y, u, v.

Proof. We make use of the "method of descent": assume that

$$(1) \qquad\qquad a^2 + b^2 = c^2 + d^2$$

$$(2) \qquad\qquad ab = 2cd$$

     (a, b, c, d positive integers such that ab is minimal).

No three of the four integers can have a common factor greater than 1. $(a, b) = 1$, because if p is any prime (2 or odd), then by equation (2), $p \mid a$, $p \mid b$ would imply that $p \mid c$ or $p \mid d$. Likewise, $(c, d) = 1$. Thus one and only one of a and b is even. Let us assume that $2 \mid b$. $a^2 + b^2$ is odd, thus, by (1), either c or d must be even; say, $2 \mid c$. This implies, by (2), that $4 \mid b$, and thus $a^2 + b^2 \equiv 1 \pmod 8$, because $a^2 \equiv 1 \pmod 8$. This again implies, by (1), that $c^2 \equiv 0 \pmod 8$, $4 \mid c$, thus $8 \mid b$. Let

$$(3) \qquad b = 2^{r+1}\overline{b}, \quad c = 2^r\overline{c} \quad (r > 1, \overline{b} \text{ and } \overline{c} \text{ odd}).$$

Equation (2) reads now

$$(4) \qquad a\overline{b} = \overline{c}d \quad (a, \overline{b}, \overline{c}, d \text{ odd}, (a, \overline{b}) = (\overline{c}, d) = 1).$$

With

$$(5) \qquad \alpha = (a, \overline{c}), \quad \beta = (a, d), \quad \gamma = (\overline{b}, \overline{c}), \quad \delta = (\overline{b}, d)$$

$\alpha, \beta, \gamma, \delta$ are relatively prime in pairs, because of (4). Thus

$$(6) \qquad a = \alpha\beta, \quad \overline{b} = \gamma\delta, \quad \overline{c} = \alpha\gamma, \quad d = \beta\delta.$$

Introducing this, together with (3), into (1), we obtain

$$\alpha^2\beta^2 + 2^{2r+2}\gamma^2\delta^2 = 2^{2r}\alpha^2\gamma^2 + \beta^2\delta^2,$$

or, with $C = 2^r\gamma$,

$$(7) \qquad 3C^2\delta^2 = (\alpha^2 - \delta^2)(C^2 - \beta^2)$$

     (C, α, β, δ relatively prime in pairs, $4 \mid C$).

It follows that $\delta^2 \mid (C^2 - \beta^2) \mid 3\delta^2$, a condition which will be satisfied if and only if

$$C^2 - \beta^2 = 3\delta^2, \quad \text{or} \quad C^2 - \beta^2 = \delta^2, \quad \text{or} \quad C^2 - \beta^2 = -3\delta^2, \quad \text{or} \quad C^2 - \beta^2 = -\delta^2.$$

Since $C^2 \equiv 0$ (mod 8), and $\beta^2 \equiv \delta^2 \equiv 1$ (mod 8), the first three equations are impossible. Thus $C^2 - \beta^2 = -\delta^2$, and therefore, by (7), $\alpha^2 - \delta^2 = -3C^2$. Combining, we obtain

(8) $$\beta^2 = \delta^2 + C^2 , \quad \text{and} \quad \beta^2 = \alpha^2 + 4C^2 .$$

By the familiar representation of irreducible Pythagorean triplets,

(9) $\qquad \beta = m^2 + n^2 , \quad C = 2mn , \quad \text{and also} \quad \beta = r^2 + s^2 , \quad 2C = 2rs .$

Thus

$$r^2 + s^2 = m^2 + n^2 , \quad rs = 2mn ,$$

and r, s, m, n are again a solution of the original Diophantine system; but

$$rs = C = 2^r \gamma \leq 2^r \overline{b} < b \leq ab ,$$

against the original assumption that ab is minimal.

Theorem 2. The system $x^2 - y^2 = u^2 + v^2$, $xy = 2uv$ has no solution in positive integers x, y, u, v.

Proof. Again we assume that

(1) $$a^2 - b^2 = c^2 + d^2$$

(2) $$ab = 2cd$$

$\qquad\qquad$ (a, b, c, d positive integers such that ab is minimal).

This time, b *must* be even, and we arrive again at equations (3), (4), (5), and (6) of the previous proof. The next equation, again with $C = 2^r \gamma$, reads now

(7) $$(\alpha^2 - \delta^2)(\beta^2 - C^2) = 5C^2 \delta^2 .$$

Thus $\delta^2 \mid (\beta^2 - C^2) \mid 5\delta^2$. Again, three of the resulting four conceivable equations are impossible (mod 8), thus $\beta^2 - C^2 = \delta^2$, and, by (7), $\alpha^2 - \delta^2 = 5C^2$. Thus

(8) $$\beta^2 = \delta^2 + C^2 , \quad \text{and} \quad \alpha^2 = \beta^2 + 4C^2 .$$

This implies again that

(9) $\qquad \beta = m^2 + n^2 , \quad C = 2mn , \quad \text{and also} \quad \beta = r^2 - s^2 , \quad 2C = 2rs .$

Thus $r^2 - s^2 = m^2 + n^2$, $rs = 2mn$, which satisfies the original Diophantine system; and $rs < ab$, against the original assumption that ab is minimal.

Theorem 3. $x^6 + y^6 = z^6$ has no solution in positive integers.

Proof. Assume that $a^6 + b^6 = c^6$ (a, b, c positive integers, $(a, b) = 1$, $3 \nmid a$).

$$a^6 = (c + b)(c - b)(c^2 + cb + b^2)(c^2 - cb + b^2) .$$

Each of the two last factors of the product is odd, and it is easily seen that each of them is relatively prime to each of the remaining three factors

(since $3 \nmid a$). Thus

$$c^2 + cb + b^2 = \beta^6 \quad (\beta \text{ odd}), \qquad c^2 - cb + b^2 = \gamma^6 \quad (\gamma \text{ odd}).$$

By addition and subtraction

$$2(c^2 + b^2) = \beta^6 + \gamma^6, \quad 2cb = \beta^6 - \gamma^6,$$

or, with $\beta^3 + \gamma^3 = 2m$, $\beta^3 - \gamma^3 = 2n$, (thus $m > 0$, $n > 0$),

$$c^2 + b^2 = m^2 + n^2, \quad cb = 2mn,$$

which is impossible by theorem 1.

Theorem 4. $x^{10} + y^{10} = z^{10}$ has no solution in positive integers.

Proof. Assume that $a^{10} + b^{10} = c^{10}$ (a, b, c positive integers, $(a, b) = 1$, $5 \nmid a$).

$$a^{10} = (c + b)(c - b)(c^4 + c^3 b + c^2 b^2 + cb^3 + b^4)(c^4 - c^3 b + c^2 b^2 - cb^3 + b^4).$$

Again, each one of the last two factors is odd, and relatively prime to each of the remaining three factors; thus these two factors are equal to β^{10} and γ^{10}, respectively. Therefore

$$2(c^4 + c^2 b^2 + b^4) = \beta^{10} + \gamma^{10}, \quad 2(c^3 b + cb^3) = \beta^{10} - \gamma^{10}.$$

With $\beta^5 + \gamma^5 = 2m$, $\beta^5 - \gamma^5 = 2n$ (thus $m > 0$, $n > 0$), this becomes

$$c^4 + c^2 b^2 + b^4 = m^2 + n^2, \quad cb(c^2 + b^2) = 2mn,$$

or finally, with $c^2 + b^2 = r$, $cb = s$,

$$r^2 - s^2 = m^2 + n^2, \quad rs = 2mn,$$

which is impossible by theorem 2.

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CONFUSION RINGS*

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In this paper,** we give a unified account of Confusion Rings, and extend the existing theory to cover the unbounded case. Previous papers on this topic are scattered throughout the literature, the results obtained being mainly of a trivial nature (see, for example, [2]); our notation and treatment are more readily generalized than that, for example, of TOLL [3] or VERWIRRT [2].

With the usual notation, two g-sets $\ominus$, $\ominus'$ are said to be *mutually confused* if there is no icosamorphism over any of their semi-loops. By a well-known result of KLOT [3], $\mathbf{G}_{(i)} \to \mathbf{G}'_{(j)}$ ($i \neq j$, i, $j = 1-5$) if and only if $\mathbf{G} \otimes \mathbf{G}' (\bmod \mathbf{I})$. We can now state the central result.***

VERWIRRT-TOLL-GOON Covering Theorem. $\ominus \leftarrow\!\!\to\ominus'$ if and only if $\ominus_{(i)}$, $\ominus'_{(j)}$ are co-heterocococomorphic over $\mathbf{I}^*$.

To save space, we make an obvious change of notation, writing $\mathbf{M}_s^{(i)}$ for h-sets; we then have $\mathbf{M}_{s,\lambda}^{(i)} \cdot \mathbf{M}_{s,\mu}^{(j)'} = 0$ if $i \neq j$ and so $\ominus$, $\ominus'$ are mutually confused for arbitrary s and $\mathbf{I}$. By DUM's Lemma, it follows readily that $\ominus_{(i)} \to \ominus_{(j)}$ for $i > j$; the result follows with the help of [7], and the extension to the case of unbounded confusion is now immediate.

REFERENCES

1. A. KLOT. Lecture notes.

2. B. VERWIRRT. Remarks on a theorem of Klot. U-Bahn-Hin-u-Zurück Karte, 1923 (out of print).

3. C. TOLL. "Z groups", S. F. Q. R (2), 17(b), 1941 (in Esperanto).

4. G. MATTHEWS. A plea for inteligibility. (Here and now).

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A NOTE ON THE FACTORIZATION OF INTEGERS

William Edward Christilles

The purpose of this paper is to demonstrate the usefulness of the binary quadratic form,

$$(1) \qquad x^2 + pqy^2 ,$$

in integers x and y with integral coefficients, in the factorization of odd integers of the form $4k + 1$, where k is a positive integer.

We begin by introducing the following two lemmas.

Lemma 1. If M is a positive integer of the form $4k + 1$ and r_1 and r_2 are any pair of positive integral roots of M where $M = r_1 r_2$, there exist integers u, m, v, n, p, and q such that:

$$(i) \qquad r_1 = pu^2 + qv^2$$

$$(ii) \qquad r_2 = pm^2 + qn^2$$

Proof: Let $p = v = n = 1$

$$\therefore r_2 - r_1 = m^2 - u^2 = 4\alpha ,$$

if M is of the form $4k + 1$. Let $\alpha = B_1 B_2$, where B_1 and B_2 are any pair of positive integral roots which satisfy $\alpha = B_1 B_2$.

$$\therefore 4\alpha = 4B_1 B_2 = (B_2 + B_1)^2 - (B_2 - B_1)^2 ,$$

and consequently, for some q,

$$r_1 = (B_2 - B_1)^2 + q$$

$$r_2 = (B_2 + B_1)^2 + q ,$$

where $(B_2 - B_1) = u$, and $(B_2 + B_1) = m$.

Lemma 2. If M is a positive integer of the form $4k + 1$ and r_1 and r_2 are any pair of positive integral roots where $M = r_1 r_2$, such that:

$$(i) \qquad r_1 = pu^2 + qv^2$$

$$(ii) \qquad r_2 = pm^2 + qn^2 ,$$

then M has the form $x^2 + pqy^2$.

Proof:

$$M = r_1 r_2 = (pm^2 + qn^2)(pu^2 + qv^2) = (ump + qnv)^2 + pq(un - vm)^2 .$$

If we let $(ump + qnv) = x$ and $(un - vm) = y$, we then have the form

$$M = x^2 + pqy^2 .$$

We now state and prove the following theorem.

Theorem: If M is a positive integer of the form $4k + 1$, and r_1 and r_2 are any pair of positive integral roots of M where $M = r_1 r_2$, there exist integers $x, y, p, q, w,$ and B such that:

(2)
(i) $$M = x^2 + pqy^2$$

(ii) $$(py)^2 - 4pB(qB - x) = (wp)^2$$

moreover, there are two integers v and n where $B = vn$, such that

(3) $$M = r_1 r_2 = \left[p\left(\frac{w-y}{2v}\right)^2 + qn^2 \right]\left[p\left(\frac{w+y}{2n}\right)^2 + qv^2 \right],$$

where $\left(\dfrac{w-y}{2v}\right)$ and $\left(\dfrac{w+y}{2n}\right)$ are integers.

Proof: It follows from Lemma 1 and Lemma 2 that:

$$M = r_1 r_2 = x^2 + pqy^2 = (ump + qvn)^2 + pq(un - vm)^2 ,$$

where $x = (ump + qvn)$ and $y = (un - vm)$. Eliminating u, from the last two relations and solving the result for m, we obtain:

(4) $$m = \frac{-yp \pm \sqrt{(py)^2 + 4vp(xn - qvn^2)}}{2vp}$$

Consequently,

$$(yp)^2 - 4Bp(Bq - x) = (wp)^2 ,$$

where $B = vn$; and solving for x, we obtain:

$$x = \frac{(wp)^2 + 4pqB^2 - (yp)^2}{4pB}$$

$$\therefore M = x^2 + pqy^2 = \left[\frac{4pqB^2 + (pw)^2 - (yp)^2}{4pB} \right]^2 + pqy^2$$

$$= \left[\frac{(wp)^4 - 2(wpyp)^2 + (yp)^4}{(4pB)^2} \right] + \left[\frac{8pq(pwB)^2 + 8pq(pyB)^2}{16(pB)^2} \right] + \left[\frac{4^2 p^2 q^2 B^4}{4^2 p^2 B^2} \right]$$

$$= p^2\left[\frac{(w^2-2wy+y^2)(w^2+2wy+y^2)}{16v^2n^2}\right] + pq\left[\frac{(w^2-2wy+y^2)+(w^2+2wy+y^2)}{4}\right] + q^2v^2n^2$$

$$= p^2\left[\frac{(w-y)^2(w+y)^2}{16v^2n^2}\right] + pq\left[\frac{(w-y)^2+(w+y)^2}{4}\right] + q^2v^2n^2$$

$$= \left[p\left(\frac{w-y}{2v}\right)^2 + qn^2\right]\left[p\left(\frac{w+y}{2n}\right)^2 + qv^2\right],$$

where $\left(\frac{w-y}{2v}\right) = m$, and $\left(\frac{w+y}{2n}\right) = u$, from (4). This completes the proof of the theorem.

We further observe that by solving the relation

$$(yp)^2 - 4pB(qB-x) = (wp)^2$$

for B, we obtain:

$$(5)\qquad B = \frac{4xp \pm \sqrt{(4xp)^2 - (4^2pq)[(wp)^2 - (yp)^2]}}{4(2pq)} = \frac{x \pm \sqrt{x^2 - pq(w^2-y^2)}}{2q}$$

Consequently,

$$x^2 - pq(w^2 - y^2) = z^2$$

or:

$$(6)\qquad\qquad x^2 + pqy^2 = z^2 + pqw^2$$

This is a simple way of proving that if a number, M, be expressed in more than one distinct way in the form $x^2 + pqy^2$, for a given p and q, then the number is composite and its factors easily determined.

An example : Let $M = 10,873$. We find that

$$M = x^2 + pqy^2 = z^2 + pqw^2 = (\pm 89)^2 + 82(\pm 6)^2 = (\pm 75)^2 + 82(\pm 8)^2$$

Thus

$$pq = 82 , \quad y^2 = (\pm 6)^2 , \quad \text{and} \quad w^2 = (\pm 8)^2$$

Substituting these values in (5), we find that $B = vn = 1$. But

$$M = \left[p\left(\frac{w-y}{2v}\right)^2 + qn^2\right]\left[p\left(\frac{w+y}{2n}\right)^2 + qv^2\right],$$

from (3), and

$$m = \frac{-yp \pm \sqrt{(yp)^2 + 4(vn)px - 4pq(vn)^2}}{2vp} ,$$

from (4). Thus for $v = 1$, $n = 1$, $w = 8$, $y = 6$, $p = 1$, and $q = 82$, we have :

$$M = (83)(131) = 10,873 \ ,$$

from (3). Should we not have been able to put M in the two forms of (6) readily, we could have attempted to satisfy the two conditions of the theorem by first putting M in the form (1), and with the information from (1), attempt to satisfy the second condition by substituting successively for B, the integers $\pm 1, \pm 2, \pm 3, \cdots$, in (2) until a suitable value for B is found. If a successful B is found, we can then easily factor M.

A second example: Let $M = 19,933$. We find that

$$M = x^2 + pqy^2 = (\pm 128)^2 + 21(\pm 13)^2$$

Thus

$$pq = 21 \ , \quad x^2 = (\pm 128)^2 \ , \quad \text{and} \quad y^2 = (\pm 13)^2 \ .$$

Next we attempt to satisfy (2) by successively substituting, for B, in (2), the integers $\pm 1, \pm 2, \pm 3, \cdots$. For $B = 14$, $p = 3$, and $q = 7$, we have:

$$(yp)^2 - 4Bp(Bq - x) = (39)^2 - 168(-30) = (81)^2 = (wp)^2$$

But

from (3),

$$M = \left[p\left(\frac{w-y}{2v}\right)^2 + qn^2 \right]\left[p\left(\frac{w+y}{2n}\right)^2 + qv^2 \right] \ ,$$

$$m = \frac{-py \pm \sqrt{(yp)^2 + 4(vn)px - 4pq(vn)^2}}{2vp} \ ,$$

from (4), and $B = vn$. Thus for $v = 7$, $n = 2$, $w = 27$, $y = 13$, $p = 3$, and $q = 7$, we have:

$$M = (31)(643) = 19,933 \ , \qquad \text{from (3).}$$

This method will then quickly factor positive integers of the form $4k + 1$, whenever the absolute value of B is relatively small or whenever one is able to represent M in the two forms of (6).

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A NOTE ON p-LIKE RINGS

Adil Yaqub

1. INTRODUCTION. In a recent paper [1], A. L. Foster introduced the concept of a Boolean-like ring, which is, *by definition*, a certain *commutative* ring. In this present note, we shall show, among other things, that the commutativity of a Boolean-like ring necessarily follows from the remaining defining conditions of such a ring. Indeed, we shall be concerned with a new class of rings, which we call *p-like* rings, and which essentially include, as special cases, the Boolean rings of Stone [3], the p-rings of McCoy and Montgomery [2] and the Boolean-like rings of Foster [1]. In fact, Boolean-like rings are now easily seen to reduce to our 2-like rings ($p = 2$). For all such rings, we prove that $ab = ba$ holds for all a, b in the ring. It is note-worthy that an example of a 2-like ring (= Boolean-like ring) which is *not* a 2-ring (= Boolean ring) has already been given in [1; p. 169]. Thus p-like rings give an essential generalization of p-rings, at least when $p = 2$. The situation is also similar when $p \neq 2$ (see example in section 3). In section 3, also, we give a simple characterization of p-like rings. The methods used in the proofs are elementary and of number-theoretic nature (compare with the proofs in [4].)

2. PRELIMINARY DEFINITIONS AND LEMMAS. Let p be a fixed *prime* throughout.

Definition. A *p-like* ring A is a ring with unit* 1 such that, for every a and b in A,

(1) $$(ab)^p - ab^p - a^p b + ab = 0 ,$$

(2) $$pa = 0 \quad (p \text{ prime}) .$$

Remark : (2) asserts that the characteristic is p. This assumption may be omitted by replacing (1) and (2) by the equivalent condition

(3) $$(ab)^p - ab^p - a^p b + (p + 1)ab = 0 .$$

The equivalence is easily seen by setting $b = 1$ in (3).

We now state the following

LEMMA 1. *In a p-like ring we have*

(i) $$(a - a^p)^2 = 0 , \quad a^{p^2 - p + 2} = a^2 , \quad (a^p)^p = a^p ;$$

(ii) $$a^2 = 0 \quad and \quad b^2 = 0 \quad imply \quad ab = 0 ;$$

(iii) $$(ab)^p = a^p b^p ;$$

(iv) $$b^p = b \quad implies \ b \ is \ in \ the \ center.$$

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Proof. Setting $b = a$ in (1), we get $(a - a^p)^2 = 0$. Now, putting $b = a^{p-1} - 1$ in (1) and using $(a - a^p)^2 = 0$, we obtain

$$0 = (a^p - a)^p - a(a^{p-1} - 1)^p - a^p(a^{p-1} - 1) + (a^p - a)$$

$$= 0 - a(a^{(p-1)p} - 1) - a^{2p-1} + a^p + a^p - a$$

$$= -a^{p^2 - p + 1} - a^{2p-1} + 2a^p .$$

Hence,

$$a^{p^2 - p + 2} = 2a^{p+1} - a^{2p} = a^2 - (a - a^p)^2 = a^2 .$$

Therefore, $a^{p^2 - p + 2} = a^2$. Hence, also, $(a^p)^p = a^p$. This proves (i).

To prove (ii), we observe that, since $a^2 = 0 = b^2$, therefore, by (1), $(ab)^p + ab = 0$. But, by (1), we also have, since $a^2 = 0 = b^2$,

$$0 = \{a(b+1)\}^p - a(b+1)^p + a(b+1)$$

$$= (ab + a)^p + ab = (ab)^p + (ab)^{p-1}a + ab .$$

Hence,

$$(ab)^{p-1}a = 0 , \quad (ab)^p = 0 , \quad ab = 0 ,$$

and (ii) is proved.

To prove (iii), we observe that, by (i),

$$(a - a^p)^2 = 0 = (b - b^p)^2 .$$

Hence, by (ii),

$$0 = (a - a^p)(b - b^p) = ab - ab^p - a^p b + a^p b^p .$$

(iii) now follows at once from (1).

To prove (iv), let $E = b^{p-1}a - ab^{p-1}$. Then, $bEb = 0$, since $b^p = b$. Hence, using (iii) and $bEb = 0$, we obtain

$$-b^p = (bE)^p - b^p = b^p E^p - b^p = b^p(E - 1)^p$$

$$= (bE - b)^p = b^p E - b^p = bE - b^p .$$

Therefore,

$$0 = bE = ba - bab^{p-1} ;$$

hence,

$$(ba - ab)(b^{p-1} - 1) = 0 .$$

Now, since the characteristic is the *prime* p, the last equality gives

$$(4) \qquad (ba - ab)(b+1)(b+2) \cdots (b + (p-1)) = 0$$

Since $b^p = b$ implies $(b+1)^p = b+1$, therefore, replacing b by $b+1$ in (4) and subtracting the result from (4), we obtain

$$(ba - ab) \cdot 1 \cdot (b+2) \cdots (b + (p-1)) = 0 .$$

Repeating this process a suitable number of times, we ultimately obtain

$$(ba - ab)(p - 2) ! (b + (p-1)) = 0$$

and, finally,

$$(ba - ab)(p - 2)! \, b = 0 .$$

Hence,

$$(p - 1)! \, (ba - ab) = 0 .$$

Therefore $ba - ab = 0$ since the characteristic is the *prime p*. This completes the proof of lemma 1.

3. THE MAIN THEOREMS. We now prove the following main theorems.

THEOREM 2. *A p-like ring is commutative.*

Proof. First,

$$a = (a - a^p) + a^p .$$

But, by lemma 1 (i),

$$(a - a^p)^2 = 0 \quad \text{and} \quad (a^p)^p = a^p ,$$

and the theorem now follows at once from lemma 1 (ii) and (iv).

COROLLARY 3. *A Boolean-like ring is commutative, where a Boolean-like ring is now defined as in* [1, (D), p. 169], *except that we no longer assume commutativity as a part of the definition.*

This follows immediately by setting $p = 2$ in the above Theorem.

We are now in a position to give a characterization of p-like rings. Indeed, we have

THEOREM 4. *Let p be a fixed prime. A ring A is p-like if, and only if, the following conditions are satisfied:*

(1) *A is commutative and with unit,*

(2) *A has characteristic p,*

(3) *For any element a in A, there exist elements x, y in A, such that* $a = x + y$, *and where* $x^p = x$ *and y is nilpotent,*

(4) *The product of any two nilpotent elements in A is zero.*

Proof. Necessity. Let A be a p-like ring. Then, (1) and (2) follow immediately from the definition and from Theorem 2. To prove (3), we first observe that, for any a in A,

$$a = a^p + (a - a^p) .$$

But, by lemma 1 (i),

$$(a^p)^p = a^p \quad \text{and} \quad (a - a^p)^2 = 0 ,$$

and (3) follows by choosing $x = a^p$ and $y = a - a^p$. To prove (4), let $a^r = 0$, $b^s = 0$, where r and s are some positive integers. By lemma 1 (i),

$$a^{p^2 - p + 2} = a^2 .$$

Therefore,

$$a^{k(p^2 - p) + 2} = a^2$$

for any positive integer k. Now choose k so large that

$$k(p^2 - p) + 2 \geqq r .$$

Then

$$0 = a^{k(p^2 - p) + 2} = a^2 .$$

Similarly, $b^2 = 0$, and (4) now follows from lemma 1 (ii).

Sufficiency. Let A be a ring such that (1)-(4) are satisfied. Let a be any element in A. Then, by (3), $a = x + y$, for some elements x, y in A such that $x^p = x$, y nilpotent. Hence,

$$a - y = x = x^p = (a - y)^p = a^p - y^p = a^p ,$$

by (1), (2) and (4). Therefore $y = a - a^p$. Hence $a - a^p$ is nilpotent. Similarly, for any b in A, $b - b^p$ is also nilpotent. Therefore, by (4), we have, for any a, b in A

$$0 = (a - a^p)(b - b^p) = a^p b^p - ab^p - a^p b + ab ,$$

which is equivalent to (1) in the definition of a p-like ring, since, by hypothesis, A is commutative. The theorem now follows by observing that, by hypothesis, $pa = 0$, and A has a unit.

With Theorem 4 as motivation, we are now able to show that p-like rings do give an essential generalization of p-rings, for *any* prime p (see introduction.) Indeed, let p be an arbitrary but fixed prime, and let R be a p-ring with unit. Let x be an imaginary square root of zero. Let $A = R[x]$ be the ring obtained by adjoining x to R. Then, A is a p-like ring but not a p-ring. That A is a p-like ring could easily be shown either directly from the definition, or via Theorem 4. Clearly, A is *not* a p-ring, since it contains some nilpotent elements.

In conclusion, I wish to express my indebtedness to the referee for his valuable suggestions.

FOOTNOTE

(*) The existence of a unit is not required in [2] and [3].

REFERENCES

1. A. L. Foster, The theory of Boolean-like rings, Trans. Amer. Math. Soc., vol. 59, 1946, pp. 166-187.

2. N. H. McCoy and D. Montgomery, A representation of generalized Boolean rings, Duke Math. J., vol. 3, 1937, pp. 455-459.

3. M. H. Stone, The theory of representations of Boolean algebras, Trans. Amer. Math. Soc., vol. 40, 1936, pp. 37-111.

4. A. Yaqub, Elementary proofs of the commutativity of p-rings, American Mathematical Monthly, vol. 64, 1957, pp. 253-254.

Purdue University

DIVISION OF A POWER SERIES BY $(1-\alpha x)^n$

D. Mazkewitsch

We derive the relation of the coefficients of two successive quotients and, using the Pascal triangle, we show how the quotient may be obtained without performing the division.

Consider an infinite series

$$S(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

The quotient $S(x)/(1-\alpha x)^n$, with $n = 1, 2, 3, \cdots$, may be represented as

$$\frac{S(x)}{(1-\alpha x)^n} = (a_0 + a_1 x + a_2 x^2 + \cdots)(1-\alpha x)^{-n}$$

$$= (a_0 + a_1 x + a_2 x^2 + \cdots)(1 - \frac{(-n)\alpha x}{1} + \frac{(-n)(-n-1)}{2!}(\alpha x)^2 - + \cdots)$$

$$= (a_0 + a_1 x + a_2 x^2 + \cdots)\left[\binom{n-1}{0} + \binom{n}{1}(\alpha x) + \binom{n+1}{2}(\alpha x)^2 + \cdots\right]$$

$$= (a_0 + a_1 x + a_2 x^2 + \cdots) \sum_{k=0}^{k=\infty} \binom{n+k-1}{k}(\alpha x)^k \ .$$

The coefficient A_i^n of x^i in this development is

$$a_0 \binom{n+i-1}{i}\alpha^i + a_1 \binom{n+i-2}{i-1}\alpha^{i-1} + a_2 \binom{n+i-3}{i-2}\alpha^{i-2} + \cdots + a_j \binom{n+i-j-1}{i-j}\alpha^{i-j}$$

$$+ \cdots + a_i \binom{n-1}{0}$$

or

$$(1) \qquad A_i^n = \sum_{j=0}^{j=i} \binom{n+i-j-1}{i-j} a_j \alpha^{i-j}$$

The coefficient of x^{i-1} in the same development is

$$A_{i-1}^n = \sum_{j=0}^{i-1} \binom{n+i-j-2}{i-j-1} a_j \alpha^{i-j-1}$$

The coefficient of x^i in the development $\dfrac{S(x)}{(1-\alpha x)^{n-1}}$ is

$$A_i^{n-1} = \sum_{j=0}^{i} \binom{n+i-j-2}{i-j} a_j \alpha^{i-j} .$$

Then

$$\alpha A_{i-1}^n + A_i^{n-1} = \sum_{j=0}^{i-1} \binom{n+i-j-2}{i-j-1} a_y \alpha^{i-j} + \sum_{j=0}^{i} \binom{n+i-j-2}{i-j} a_j \alpha^{i-j}$$

$$= \sum_{j=0}^{i} \binom{n+i-j-1}{i-j} a_j \alpha^{i-j}$$

$$= A_i^n$$

Hence

(2) $$A_i^n = \alpha A_{i-1}^n + A_i^{n-1}$$

Relation (2) shows that the c o e f f i c i e n t of a power x^i in an expansion $\dfrac{S(x)}{(1-\alpha x)^n}$ is equal to α times the coefficient of the power to the left plus the coefficient of the same power x^i in the expansion $\dfrac{S(x)}{(1-\alpha x)^{n-1}}$.

From (1) we see that the numerical factors of the coefficients of x^i are the coefficients of the n-th diagonal in the Pascal triangle.

This law enables us to write down the quotient of a power series by $(1-\alpha x)^n$ without performing the operation of division. Example. Find $\dfrac{e^x}{(1-x)^4}$, which may be written

$$\frac{\left(1 + \dfrac{x}{1} + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \cdots\right)}{(1-x)^4} .$$

Here we have:

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 1/2, \quad a_3 = 1/6, \quad \cdots, \quad \alpha = 1$$

$$\therefore \frac{e^x}{(1-x)^4} = 1 + (4 \cdot 1 \cdot 1 + 1)x + (10 \cdot 1 \cdot 1^2 + 4 \cdot 1 \cdot 1 + \tfrac{1}{2})x^2$$

$$+ (20 \cdot 1 \cdot 1^3 + 10 \cdot 1 \cdot 1^2 + 4 \cdot \tfrac{1}{2} \cdot 1 + \tfrac{1}{6})x^3 + \cdots$$

$$= 1 + 5x + \frac{29}{2} x^2 + \frac{193}{6} x^3 + \cdots$$

If the divisor is $(\beta - \alpha x)^n$, we reduce to the previous case by writing

$$(\beta - \alpha x)^n = \beta^n (1 - \frac{\alpha}{\beta} x)^n.$$

α in the shown expansion is replaced by $\frac{\alpha}{\beta}$.

University of Cincinnati
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The axiomatic method consists simply in making a complete collection of the basic concepts as well as the basic facts from which all concepts and theorems of a science can be derived by definition and deduction respectively. If this is possible, then the scientific theory in question is said to be definite according to Husserl. Such is the case for the theory of space. Of course, from the axioms of geometry I cannot possibly deduce the law of gravitation. Hence it was necessary to explain above what is to be considered a pertinent proposition of a given field of inquiry. Similarly the axioms of geometry fail to disclose whether Zurich is farther from Hamburg than Paris. Though this question deals with a geometrical relation, the relation is one between individually exhibited locations. Thus, precisely speaking, what is supposed to be deducible from the axioms are the pertinent general true propositions.
Reprinted with permission from Hermann Weyl, *Philosophy of Mathematics and Natural Science*, Princeton University Press, Princeton, 1949.

CURRENT PAPERS AND BOOKS

Edited by H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department and all books sent for review in the MATHEMATICS MAGAZINE should be sent to *Professor Homer V. Craig, 3104 Grandview Street, Austin 5, Texas.*

BOOKS RECEIVED FOR REVIEW

Algebra and Trigonometry. By Edward H. Cameron. Henry Holt and Company, New York, 1960, xi + 290 pp., $5.00.

Modern Mathematics. By Samuel I. Altwerger. The Macmillan Company, New York, 1960, xii + 462 pp., $6.75.

Mathematics in Action. By O. G. Sutton. Harper and Brothers, New York, Second Edition 1960, xvi + 236 pp., $1.45.

From Euclid to Eddington. By Edmund T. Whittaker. Dover Publications, Inc., New York, 1958, ix + 212 pp., $1.35.

Elementary Statistics. By Sidney F. Mack. Henry Holt and Company, New York, 1960, ix + 198 pp. $4.50.

Differential and Integral Calculus. By James R. F. Kent. Houghton Mifflin Company, Boston, 1960, xv + 511 pp., $6.75.

Elementary Algebra. By Donald S. Russell. Allyn and Bacon, Inc., Boston, 1959, ix + 297 pp.

Practical Problems in Mathematics of Finance. By Robert Cissell and Helen Cissell. Houghton Mifflin Company, Boston, 1960, vi + 86 pp., $1.80.

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PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

411. *Proposed by C. W. Trigg, Los Angeles City College.*

Each letter in this cryptarithm uniquely represents a digit. Reconstruct the factorization.

$$
\begin{array}{r|rrrr}
E & R & T & M & P \\
\hline
& L & B & T & O \\
\hline
& & I & T & S \\
\hline
& & & R & L
\end{array}
$$

[Dedicated to RTMP 0123456789]

412. *Proposed by D. Moody Bailey, Princeton, West Virginia.*

P is any point on the circumcircle of triangle ABC. Rays from B and C through P meet CA and AB at points E and F respectively. Considering the segments involved as directed quantities, show that

$$\frac{b^2}{a^2} \cdot \frac{BF}{FA} + \frac{c^2}{a^2} \cdot \frac{CE}{EA} = -1,$$

where a, b, and c are the sides opposite the vertices A, B, and C of triangle ABC.

413. *Proposed by the late Victor Thébault, Tennie, Sarthe, France.*

If a, b, c, d, e, and f are consecutive terms of the Fibonacci Series; 1, 1, 2, 3, 5, 8, $\cdots$, prove that

$$
\begin{vmatrix}
a & b & x-a-b \\
b & c & x-b-c \\
c & d & x-c-d
\end{vmatrix}
\times
\begin{vmatrix}
b & c & x-b-c \\
c & d & x-c-d \\
d & e & x-d-e
\end{vmatrix}
=
\begin{vmatrix}
c & d & x^2-c-d \\
d & e & x^2-d-e \\
e & f & x^2-e-f
\end{vmatrix}
$$

414. *Proposed by Sidney Kravitz, Dover, New Jersey.*

If the calculation $f(x) + g(y) = h(z)$ is to be performed, it is possible to construct a slide rule for the purpose. Show that the following formulas lend themselves to calculation on a special slide rule:

(1)
$$z = x + xy + y$$

(2)
$$z = \frac{x+y}{1+xy}$$

415. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Prove

$$\sum_{p=0}^{n} \binom{n}{p} \cos(p)x \sin(n-p)x = 2^{n-1} \sin nx .$$

416. *Proposed by Barney Bissinger, Lebanon Valley College, Pennsylvania.*

Evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n} \int_0^\infty \frac{e^{-x} x^n}{(n-k)! \, n^k} dx .$$

417. *Proposed by Monte Dernham, San Francisco, California.*

The proverbial messenger, whose favorite passtime is to ride from the rear of a marching column to the front and back to the rear, has been at it again. On a recent occasion, a column x miles long advanced y miles while the messenger was thus motorcycling. On a similar occasion, when he happened to be on horseback, a column two-thirds as long advanced three times as far. After figuring a bit, the messenger was surprised to discover that on each occasion he had traveled exactly y^2 miles. Assuming all speeds were uniform, find x and y.

SOLUTIONS

Late Solutions

382. *William E. F. Appuhn, St. John's University, New York.*
383, 384, 385, 386, 387, 388, 389. *Melvin Hochster, Stuyvesant High School, New York.*
369. *C. F. Pinzka, University of Cincinnati.*

Erratum

In problem 402, January 1960, page 166, the equation should read

$$F(x) = \sum_{n=k}^{\infty} \binom{n}{k} x^n = x^k (1-x)^{-k-1} .$$

The Hypocycloid

382. [May 1959 and January 1960] *Proposed by C. N. Mills, Sioux Falls College, South Dakota.*

For the hypocycloid

$$x^{2/3} + y^{2/3} = a^{2/3} ,$$

(α, β) is the center of curvature. Show that

$$\alpha + \beta = (x^{1/3} + y^{1/3})^3 .$$

Alternate solution by Robert Kilmoyer, Lebanon Valley College, Pennsylvania. Superimposing an $\vec{i}, \vec{j}$ vector system, we determine from the equivalent parametric equations,

$$x = a\cos^3 t, \quad y = a\sin^3 t ,$$

the unit normal vector at (x, y) or t to be

$$\vec{n} = (\sin t)\vec{i} + (\cos t)\vec{j} .$$

Now the vector $\vec{c} = \alpha\vec{i} + \beta\vec{j}$ is by definition

$$\vec{c} = \rho\vec{n} + \vec{r} ,$$

where ρ is the radius of curvature and $\vec{r}$ is the position vector for (x, y). Since $\rho = 3a\sin t\cos t$ and $\vec{r} = (a\cos^3 t)\vec{i} + (a\sin^3 t)\vec{j}$, we have

$$\vec{c} = a[\cos^3 t + 3\cos^2 t\sin t]\vec{i} + a[\sin^3 t + 3\sin^2 t\cos t]\vec{j} ,$$

so

$$\alpha + \beta = a(\cos t + \sin t)^3 = (x^{1/3} + y^{1/3})^3 .$$

Regular Polyhedra

390. [November 1959] *Proposed by C. W. Trigg.*

A regular tetrahedron and a regular octahedron have equal edges. Find the ratio of their volumes without computing the volume of either.

Solution by Leon Bankoff, Los Angeles, California. The dissection of a regular tetrahedron by planes through the midpoints of the four triads of edges issuing from common vertices produces four miniature regular tetrahedra whose edges are equal to those of the residual regular octahedron.

The ratio of volumes of the large to one of the small tetrahedra is $8:1$ because the ratio of their edges is $2:1$. Consequently the total volume of the four small tetrahedra is equal to that of the octahedron, and the required ratio is $1:4$.

Also solved by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; John Reed, Raytheon Co., Wayland, Massachusetts; Dale Woods, State Teachers College, Kirksville, Missouri; and the proposer.

Limit of a Quotient

391. [November 1959] *Proposed by Melvin Hochster, New York.*

If

$$S_k = \sum_{i=0}^{2^{k-1}} \binom{2^k}{2i} n^{(2^{k-1}-i)}$$

and

$$T_k = \sum_{i=1}^{2^{k-1}} \binom{2^k}{2i-1} n^{(2^{k-1}-i)}$$

prove that the limit of the sequence S_k/T_k as $k \to \infty$ is $\sqrt{n}$ when $n > 0$.

Solution by Chih-yi Wang, University of Minnesota.

Since

$$S_k + \sqrt{n}T_k = n^{2^{k-1}}(1 + n^{-1/2})^{2^k}$$

$$S_k - \sqrt{n}T_k = n^{2^{k-1}}(1 - n^{-1/2})^{2^k}$$

we have

$$S_k = [(1 + n^{-1/2})^{2^k} + (1 - n^{-1/2})^{2^k}]n^{2^{k-1}}/2$$

$$T_k = [(1 + n^{-1/2})^{2^k} - (1 - n^{-1/2})^{2^k}]n^{2^{k-1}}/2\sqrt{n}$$

and the desired result follows from the fact that

$$(1 - n^{-1/2})^{2^k} \to 0 \quad \text{as} \quad h \to \infty \quad \text{for} \quad n > 0.$$

Also solved by Steve Andrea, Oberlin, Ohio; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; and the proposer.

An Error in Logarithms

392. [November 1959] *Proposed by Chih-yi Wang.*

A student used the incorrect formula $\log_e(m+n) = (\log_e m)(\log_e n)$ instead of the $\log_e(mn) = (\log_e m) + (\log_e n)$.

a). Show that the equation he used has at least one solution in n and m.

b). Are there general solutions of the equation $\log_e(x+y) = (\log_e x)(\log_e y)$?

I *Solution by Dale Woods, State Teachers College, Kirksville, Missouri.*

a) Let $n = 2e$, then $m = \dfrac{(1 + \sqrt{1+8e})}{2}$ can be shown to be a solution.

b) Let $y = ke$, $k > 2$. Substitution of this value of y in the equation yields, after simplification, $x^k - x - ke = 0$. For a given $k > 2$,

$$f(x) = x^k - x - ke = 0$$

for some x; since $f(x) > 0$ for $x = ke$, $f(x) < 0$ for $x = (1/k)^{1/(k-1)}$, and $f(x)$ is continuous for $x > 0$. Hence there are general solutions of the equation.

II *Solution by Michael J. Pascual, Siena College, New York.*

a) $\ln(m+n) = \ln m \ln n$; setting $m = n$ we find

$$\ln 2n = (\ln n)^2$$
$$(\ln n)^2 - \ln n - \ln 2 = 0$$
$$\ln n = \frac{1 \pm \sqrt{1 + 4\ln 2}}{2}$$

$$n = e^{\frac{1 \pm \sqrt{1+4\ln 2}}{2}}$$

b) $$\ln(x+y) = \ln x \ln y \; ;$$

setting $y/x = r$ or $y = rx$

$$\ln(x+rx) = \ln x \ln rx$$

$$\ln x + \ln(1+r) = \ln x (\ln r + \ln x)$$

$$(\ln x)^2 + (\ln r - 1)\ln x - \ln(1+r) = 0$$

$$\ln x = \frac{1 - \ln r \pm \sqrt{(\ln r - 1)^2 + 4\ln(1+r)}}{2}$$

$$x = \exp \frac{1 - \ln r \pm \sqrt{(\ln r - 1)^2 + 4\ln(1+r)}}{2}$$

$$y = x \exp \frac{1 - \ln r \pm \sqrt{(\ln r - 1)^2 + 4\ln(1+r)}}{2}$$

Also solved by D. A. Breault (Part a), Sylvania Electric Products Inc., Waltham, Massachusetts; William Squire, Southwest Research Institute, San Antonio, Texas; and the proposer (Part a).

Series-Product Equivalence

393. [November 1959] *Proposed by M. S. Klamkin, AVCO, Lawrence, Massachusetts.*

Find a power series expansion of

$$P = \prod_{r=1}^{\infty} (1 + x^{2^r}) \quad \text{for } |x| < 1 .$$

Solution by Chih-yi Wang, University of Minnesota.
Define

$$P_N = \prod_{r=1}^{N} (1 + x^{2^r})$$

Then

$$P_N = \frac{(1-x^2)P_N}{1-x^2} = \frac{1 - x^{2^{N+1}}}{1-x^2}$$

And

$$P = \lim_{N \to \infty} P_N = (1-x^2)^{-1} = \sum_{n=0}^{\infty} x^{2n} \quad \text{for } |x| < 1 .$$

Also solved by D. A. Breault, Sylvania Electric Products Inc., Waltham, Massachusetts; R. G. Buschman, Oregon State College; and the proposer.

Buschman gave as a reference Knopp, Infinite Series (2nd English Edition), P. 436, Example 3.

Convergent Subseries

384. [November 1959] *Proposed by Joseph Andrushkiw, Seton Hall University.*

Striking out a number of terms in the harmonic series without changing its order leaves a subseries (finite or infinite). Show that if r is any positive real number, there exist infinitely many subseries convergent to r.

Solution by Benjamin L. Schwartz. We prove a slightly more general result. Let $S = \Sigma x_n$ be a divergent series of positive terms such that $\lim x_n = 0$. Let r be an arbitrary positive number. Then there exists a subseries S^1 of S that converges to r. A corollary (which we also prove) is that there are infinitely many such subseries convergent to r.

Let k_1 be any natural number such that $x_{k_1} < r$. Such a k_1 exists by virtue of the hypothesis that $\lim x_n = 0$. Now note that $\sum_{n=k_1}^{\infty} x_n$ diverges to $a + \infty$ since it differs from the divergent series S by only a finite number of terms. Hence, for sufficiently large k we have $r - \sum_{n=k_1}^{k} x_n < 0$. But since $r - x_{k_1} > 0$, we can be certain that there is a largest integer k_2 (say) such that $r - \sum_{n=k_1}^{k_2} x_n > 0$. Let $r_1 = r - \sum_{n=k_1}^{k_2} x_n$.

Continue inductively in the same manner. Choose k_{2i+1} as any sufficiently large integer that $k_{2i+1} > k_{2i}$ and $x_{k_{2i+1}} < r_i$. Define k_{2i+2} as the largest integer such that $r_i - \sum_{n=k_{2i+1}}^{k_{2i+2}} x_n > 0$. Define $r_{i+1} = r_i - \sum_{n=k_{2i+1}}^{k_{2i}} x_n$. Then the series

$$S^1 = \sum_{k_1}^{k_2} x_n + \sum_{k_3}^{k_4} x_n + \sum_{k_5}^{k_6} x_n + \cdots = \sum_{i=1}^{\infty} \sum_{n=k_{2i-1}}^{k_{2i}} x_n$$

is bounded above by r. Furthermore, because of the way the k_{2i} are defined, we have $r - \sum_{i=1}^{p} \sum_{n=k_{2i-1}}^{k_{2i}} x_n < x_{k_{2_p}+1}$ which can be made arbitrarily small since $\lim x_n = 0$. It is clear from the method of construction of S^1, particularly the k_{2i+1}, that the construction can be effected in infinitely many ways each leading to distinct series S^1.

Also solved by the proposer.

Prime Factors

395. [November 1959] *Proposed by Sidney Kravitz, Dover, New Jersey.*

It is well known that $f(n) = n^2 - n + 41$ yields prime numbers for $n \leq 40$. Show that $f(n)$ contains at most two prime factors for $n \leq 420$.

$\mathbf{I}$ *Solution by Dale Woods, State Teachers College, Kirksville, Missouri.* Since $f(n) = n^2 - n + 41$ is a prime unless $n = 41k$ or $n - 1 = 41k$ we only need to show $41k^2 \pm k + 1$ is a prime for $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$. This may be verified by any table of primes. Hence $f(n)$ is the product of at most two primes for $n \leq 420$.

$\mathbf{II}$ *Solution by D. A. Breault, Sylvania Electric Products Inc., Waltham, Massachusetts.* Using an IBM 709 calculator the values of n, $f(n) = n^2 - n + 41$, and the prime factors of $f(n)$ were tabulated. For the values of n from 41 to 450 this table revealed that $f(n)$ has two prime factors for all n except 421 and 432. Both $f(421)$ and $f(432)$ have three prime factors.

This calculation occupied the IBM 709 for approximately five minutes. *Also solved by the proposer.*

Sum of Greatest Integers

396. [November 1959] *Proposed by J. B. Love, Eastern Baptist College, Pennsylvania.*

Show that $\sum_{j=0}^{n-1} [x + j/n] = [nx]$ where the brackets denote the greatest integer function.

$\mathbf{I}$ *Solution by W. O. J. Moser, University of Manitoba.* Since $[x] \leq x < [x] + 1$, it follows that $[x] + (k/n) \leq x < [x] + \{(k+1)/n\}$ where k is one of the integers $0, 1, 2, \cdots, n-1$. Thus

$$[x + j/n] = \begin{array}{ll} [x] \quad, & j = 0, 1, 2, \cdots, n-k-1, \\ [x]+1, & j = n-k, n-k+1, \cdots, n-1, \end{array}$$

and

$$\sum_{j=0}^{n-1} [x + j/n] = (n-k)[x] + k([x]+1) = n[x] + k.$$

Also,

$$n[x] + k \leq nx < n[x] + k + 1,$$

So that $[nx] = n[x] + k$.

$\mathbf{II}$ *Solution by R. G. Buschman, Oregon State College.* Let $\theta = x - [x]$ and note that

$$(1) \qquad [nx] = n([x] + \theta) = n[x] + n\theta = n[x] + [n\theta],$$

$$(2) \sum_{j=0}^{n-1} [x+j/n] = \sum_{j=0}^{n-1} [x] + \theta + j/n = \sum_{j=0}^{n-1} [x] + [\theta + j/n] = n[x] + \sum_{j=0}^{n-1} [\theta + j/n].$$

Hence it is sufficient to prove the relation in the case $x = \theta$, $0 \leq \theta < 1$. Consider $u = f(j) = \theta + j/n$ for $0 \leq x < 1$, $0 \leq j \leq n-1$. Now $u = 1$ for $j = n - n\theta$ so that $[\theta + j/n] = 0$ unless $j \geq n - n\theta$, i. e. $j > n - n\theta - 1$. Therefore

$$\sum_{j=0}^{n-1} [\theta + j/n] = \sum_{n-n\theta-1 < j \leq -1} 1 = [(n-1) - (n-n\theta-1)] = [n\theta].$$

MATHEMATICS MAGAZINE

Also solved by D. A. Breault, Sylvania Electric Products Inc., Waltham, Massachusetts; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; Sidney Kravitz, Dover, New Jersey; Michael J. Pascual, Siena College, New York; William Squire, Southwest Research Institute, San Antonio, Texas; and the proposer.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 266. If p is a prime number greater than 3, then $p^2 + 2$ is composite. [*Submitted by Huseyin Demir*]

Q 267. Prove that the average of velocity with respect to distance is greater than or equal to the average of velocity with respect to time. [*Submitted by M. S. Klamkin*]

Answers

A 266. If p is a prime exceeding 3 then we have $p^3 + 2 = (6m \mp 1)^3 + 2 \equiv 0 \pmod 3$.

A 267. To prove $\overline{V}_s \geqq \overline{V}_t$, we have

$$\frac{\int_{s_2}^{s_1} V \, ds}{\int_{s_2}^{s_1} ds} \geqq \frac{\int_{t_2}^{t_1} V \, dt}{\int_{t_2}^{t_1} dt}$$

but this leads to

$$\frac{\int_{t_2}^{t_1} V^2 \, dt}{\int_{t_2}^{t_1} V \, dt} \geqq \frac{\int_{t_2}^{t_1} V \, dt}{\int_{t_2}^{t_1} dt}$$

which follows from the Cauchy-Schwarz Inequality.

VICTOR THÉBAULT
1882 - 1960

March 19, 1960 marked the end of the fruitful career of a distinguished member of the editorial staff of MATHEMATICS MAGAZINE, Victor Michel Jean-Marie Thébault. Mercifully, his death closely followed a stroke which greatly impaired his mobility and speech. He is survived by his wife, five sons and a daughter.

Born on March 6, 1882 at Ambrieres-le-Grand (Mayenne), France, he early demonstrated an ability which led to a scholarship at the teacher's college in Laval, where he studied from 1898 to 1901. After graduation, he taught at Pré-en-Pail from 1902 to 1905, when he obtained a professorship at the technical school of Ernée (Mayenne). In 1909, a first place in competitive examination led to his certificate of capacity for a scientific professorship in teachers' colleges.
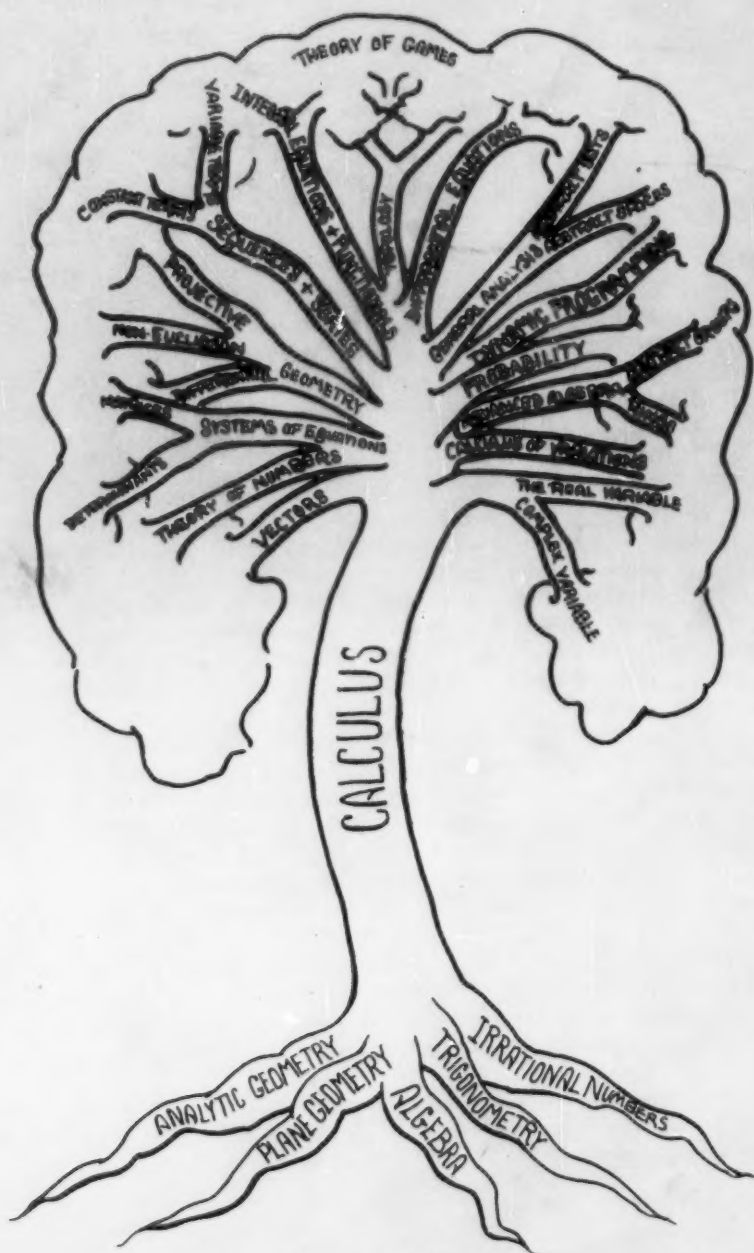
Finding the salary of a professor inadequate to support his large family, M. Thébault left teaching to act as a factory superintendent at Ernée from 1910 to 1923. Following this, he became Chief Insurance Inspector at Le Mans (Sarthe) in 1924, which position he held until 1940 when he retired to live in Tennie (Sarthe).

M. Thébault's interest and activity in mathematics continued unabated after he abandoned active teaching. His keen imagination and penetrating insight led him to many interesting discoveries in the fields of number theory and geometry. His originality produced hundreds of memoirs and articles in the modern geometry of the triangle and tetrahedron which have appeared in many magazines throughout the world. At the time of his demise he was engaged in the preparation of a brochure on the arbelos (The Shoemaker's Knife).

A firm believer in the value of the challenge problem, M. Thébault contributed thousands of original problems and solutions to the various mathematical publications which carry problem sections. His contributions to the problem departments of the *American Mathematical Monthly* alone exceeded 600.

Recognition of his ability and performance led the French Government in 1932 to make him an Officier de L'Instruction Publique. In 1935 he was made a Chevalier de l'Ordre de la Couronne de Belgium.

C. W. Trigg